GLOBAL DYNAMICS ON ONE-DIMENSIONAL EXCITABLE MEDIA

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Abstract. The FitzHugh–Nagumo system has been studied extensively for several decades. It has been shown numerically that pulses are generated to propagate and then some of the pulses are annihilated after collision. For the mathematical understanding of these complicated dynamics, we investigate the global dynamics of a one-dimensional free boundary problem in the singular limit of a FitzHugh–Nagumo type reaction–diffusion system. By introducing the notion of symbolic dynamics, we show that the asymptotic behaviors of solutions are classified into three categories: (i) the solution converges uniformly to the resting state; (ii) the solution converges to a series of traveling pulses propagating in either the same direction or both directions; and (iii) the solution converges to a propagating wave consisting of multiple traveling pulses and two traveling fronts propagating in both directions.

1. Introduction

Various patterns are observed in the natural world, such as crystallization [2], bacterial colonies [3], and chemical reactions [15]. Many researchers have been attracted by the problem of pattern formation, and they have proposed mathematical models to study these patterns. Among them, excitable systems provide many interesting patterns, such as traveling pulses, spiral waves, target patterns, and more complicated dynamics, which may arise in reaction–diffusion systems such as the Belousov–Zhabotinsky reaction, and the FitzHugh–Nagumo system (e.g. see [1, 13, 16, 20, 22, 23]). These patterns have been observed by experiments and numerical simulations. Mathematically, there are several ways to capture the patterns of the solutions of reaction–diffusion systems: phase analysis [21], bifurcation theory [13], and interface analysis [17, 19]. However, it is still difficult to derive the global dynamics of solutions of reaction–diffusion systems. To study the global dynamics, in this study, we focus on the global dynamics of existing pulses and neglect the nucleation of pulses.

To introduce our model, we review the singular limit problem obtained from reaction–diffusion systems. Singular limit analysis is often used to obtain the equation for the sharp interface between coexisting different chemical or physical states. Indeed, it is well known that many reaction–diffusion systems can produce interfaces when some diffusion rates are small enough or some reaction terms are large enough. A typical example is the following FitzHugh–Nagumo reaction–diffusion system:

$$
\begin{cases} u_t = \varepsilon\Delta u + \dfrac{1}{\varepsilon}(F(u)-v), \\ v_t = G(u,v), \quad x\in\mathbb{R}^n,\ t>0, \end{cases} \tag{1.1}
$$

where nonlinear functions are given by

$$
F(u)=u(1-u)(u-a), \quad G(u,v)=\gamma_0+\gamma_1 u-\gamma_2 v
$$

for some $a\in(0,1)$ and $\gamma_i\geq 0$, $i=0,1,2$. For each $v\in(\min_{u\in[0,1]}F(u),\max_{u\in[0,1]}F(u))$, $F(u)-v=0$ has three zeros, which are denoted by $h_\pm(v)$ and $h_0(v)$ with $h_-(u)<h_0(v)<h_+(u)$, where $h_\pm(v)$

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are stable equilibria for the ODE: $u_t = F(u) - v$. By a formal analysis (see [11, 22] for more details), the singular limit problem of (1.1) as $\varepsilon \downarrow 0$ can be written as

$$\begin{cases} V_n &= R(v) \quad \text{in } \partial\Omega(t), \\ v_t &= G\Big(\mathbf{1}_{\Omega(t)} h_+(v) + (1 - \mathbf{1}_{\Omega(t)}) h_-(v), v\Big), \end{cases}$$

where $\Omega(t)$ is the region on which u converges to $h_+(v)$ as $\varepsilon \downarrow 0$, V_n is a normal velocity of $\Omega(t)$, and $R(v)$ is the speed of one-dimensional traveling waves of $u_t = u_{xx} + F(u) - v$. To regularize the system, curvature κ is often added at x of $\partial\Omega(t)$, and then we have

$$\begin{cases} V_n &= R(v) - \varepsilon\kappa \quad \text{in } \partial\Omega(t), \\ v_t &= G(h_+(v)\mathbf{1}_{\Omega(t)} + h_-(v)(1 - \mathbf{1}_{\Omega(t)}), v) \end{cases} \tag{1.2}$$

with a small positive constant ε, where $\mathbf{1}_{\Omega(t)}$ is a characteristic function of $\Omega(t)$ in $\mathbb{R}$. To the best of our knowledge, this type of systems was introduced by Fife [11] in 1984 (therein, the v-equation contains a diffusion term). These types of interface problems may involve abundant spatio-temporal patterns and bring mathematical challenges. The problem (1.2) with diffusion term in the v-equation has been studied in the literature. For example, the existence of solutions was studied in [4, 5]. The global existence of a weak solution was established in [12] by the viscosity solution approach. The well-posedness for a one-dimensional space case and $\varepsilon = 0$ was studied in [14]. In fact, the diffusion term that appears in the v-equation can guarantee the regularity of solutions. However, the case without diffusion term in the equation for v has not been studied so much. In the one-dimensional space case, the existence, uniqueness, and non-uniqueness of weak solutions were considered in [6], but the global dynamics is still not understood well because of the complexity. It is rather difficult to study the global dynamics such as the asymptotic behavior of solutions for these types of interface problems with initial data. Hence, one may study a simplified model to get a better understanding of the complex dynamics that has been observed in the experiments and simulations (see, e.g. [1]).

In this paper, we consider the following FitzHugh–Nagumo type reaction–diffusion system proposed in [7]:

$$\begin{cases} u_t &= \Delta u + \dfrac{1}{\varepsilon^2}(f_\varepsilon(u) - \varepsilon\beta v), \\ v_t &= g(u, v), \quad x \in \mathbb{R}^2,\ t > 0, \end{cases} \tag{1.3}$$

where $\varepsilon > 0$ and

$$f_\varepsilon(u) := u(1-u)\left(u - \frac{1}{2} + \varepsilon\alpha\right), \quad g(u, v) = g_1 u - \frac{g_2 v}{g_3 v + g_4}$$

for some $g_j > 0$ for $j = 1, 2, 3, 4$. For a solution $(u^\varepsilon, v^\varepsilon)$ of (1.3), we can expect that as $\varepsilon \downarrow 0$, u^ε converges to 1 or 0 from the first equation of (1.3). In this case, we denote the region where u^ε converges to 1 by $\Omega(t)$. Under the assumption

(A) $g_1 g_3 > 2g_2$,

it was shown in [7] that by a singular limit analysis (taking $\varepsilon \downarrow 0$), (1.3) is reduced to

$$\begin{cases} V &= W(v) - \kappa, \qquad (x, y) \in \partial\Omega(t),\ t > 0, \\ v_t &= g(\mathbf{1}_{\Omega(t)}, v), \qquad (x, y) \in \mathbb{R}^2,\ t > 0, \end{cases} \tag{1.4}$$

where κ is the curvature function of $\partial\Omega(t)$, V is the outer normal velocity of $\partial\Omega(t)$, and

$$W(v) = a - bv, \quad a = \sqrt{2}\alpha, \quad b = 6\sqrt{2}\beta.$$

For further details refer to [7, Appendix].

The existence of traveling spots of system (1.4) was established in [7]. The authors in [8] showed that the traveling spot converges to a planar traveling wave when some system parameter varies. The existence, uniqueness, and stability of traveling curved waves was reported in [18], where each interface was represented by a graph.

To get a better understanding of the complicated dynamics of (1.4), one may consider the one-dimensional spatial problem, where the curvature effect vanishes completely. This leads us to study the following problem:

$$\begin{cases} V &= W(v(x,t)), \qquad x \in \partial\Omega(t),\ t>0, \\ v_t &= g(\mathbf{1}_{\Omega(t)}, v), \qquad x \in \mathbb{R},\ t>0. \end{cases} \tag{1.5}$$

Throughout the paper, we assume condition **(A)**, which is necessary for the singular limiting process (see [7]). In fact, in our analysis, it suffices to assume a weaker condition $g_1 g_3 > g_2$.

Our aim is to establish the complete global dynamics of (1.5) with suitable initial data. Here, we assume that $\Omega(0) := \Omega_0$ consists of m disjoint bounded intervals, i.e.,

$$\Omega_0 := \bigcup_{j=1}^{m} (x^0_{2j-1}, x^0_{2j}).$$

Under suitable conditions, we can show that there exists $x = x_k(t)$, $k = 1, ..., 2m$ satisfying $x_k(0) = x^0_k$ and certain ODEs such that

$$\Omega(t) := \bigcup_{j=1}^{m} (x_{2j-1}(t), x_{2j}(t)) \tag{1.6}$$

for some time interval. Here, each $x = x_k(t)$ is said to be an interface of (1.5). From a modeling viewpoint, we call each $(x_{2j-1}(t), x_{2j}(t))$ $(j = 1, ..., m)$ an *excitation interval* at time t; $\Omega(t)$ the *excitation area* at time t; $\mathbb{R}\backslash\Omega(t)$ the *resting area* at time t.

Note that interfaces may collide with each other such that a classical solution cannot be extended beyond this time, called the *annihilation time*. Such a phenomenon was already considered by Chen and Gao [6] who studied problem (1.2). Based on some suitable setting, they introduced the notion of a "switching" solution of (1.2), namely, a solution switching from excitation area to resting area or from resting area to excitation area at some time, to study nucleation and annihilation. Though they call it a weak solution, it is different from the usual sense of a weak solution. To avoid the confusion, we call it a "switching" solution. They showed the uniqueness of the "switching" solution (see [6] for the details). Among other things, they also discussed the ill-posed issue and studied the nucleation phenomenon of interfaces which will not be investigated in our study. The global dynamics of (1.2) is still open.

To study the global dynamics of our system, it is required to study the continuation of solutions beyond the annihilation time. In [9], we introduce the notion of weak solutions (see Definition 2.4 below), and establish the global existence and uniqueness of weak solutions to (1.5) with initial data. In this paper, we aim to understand the global dynamics of (1.5) with suitable initial data. Because of the presence of the annihilation, it is not easy to characterize the long-time behavior of solutions of (1.5) when the number of the initial excitation intervals are large. To overcome this, we adopt the notion of symbolic dynamics (cf.[10]) to determine the eventual number of excitation intervals depending on the sign of $W(v_0(x))$ on $\partial\Omega(0)$. More precisely, we show that the number of excitation intervals is non-increasing in time and eventually becomes a constant such that the convergence of solutions to multiple traveling pulses (fronts) can be established completely. A key idea to the proof of the convergence results is to introduce the so-called arrival time of interfaces (see Definition 2.3 below), which helps us establish important estimates.

The rest of the paper is organized as follows. In Section 2, we recall the definition of classical and weak solutions for our model introduced in [9], and then present the main results. In Section 3, we investigate the global dynamical behavior of solutions to problem (1.5).

2. Main results

In this section, we state our main results. A solution (Ω, v) of (1.5) is a (unbounded) traveling front solution with speed c if

$$\Omega(t) = \Omega_F(t) := \{x \mid x < ct\}, \quad v(x,t) = \varphi_F(x - ct),$$

where function $\varphi_F(\cdot) \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ satisfies $\varphi_F(-\infty) = \infty$ and $\varphi_F(+\infty) = 0$. By a traveling pulse solution with the speed c, we mean that a solution (Ω, v) of (1.5) takes the following form:

$$\Omega(t) = \Omega_P(t) := \{x \mid ct - \ell_P < x < ct\}, \quad v(x,t) = \varphi_P(x - ct)$$

for some positive constant ℓ_P, and a function $\varphi_P(\cdot) \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0, -\ell_P\})$ satisfying $\varphi_P(\pm\infty) = 0$.

Our first main result shows the existence of a traveling front solution and a traveling pulse solution of (1.5) in terms of the following functions used in [7, 8, 18]:

$$G_0^{-1}(v) := \int_M^v \frac{d\xi}{g(0,\xi)}, \quad G_1^{-1}(v) := \int_0^v \frac{d\xi}{g(1,\xi)}, \tag{2.1}$$

where M is given in (2.4) below.

By simple computation, we have

$$0 < g_1 - \frac{g_2}{g_3} \le G_1'(\xi) \le g_1 \quad \text{for } \xi \ge 0. \tag{2.2}$$

Theorem 2.1. *There exist a traveling front solution $(\Omega_F(t), \varphi_F(x - at))$ and a traveling pulse solution $(\Omega_P(t), \varphi_P(x - at))$ of (1.5) with speed a, where*

$$\Omega_F(t) := \{x \mid x < at\}, \quad \varphi_F(z) := G_1\left(\frac{(-z)^+}{a}\right),$$

$$\Omega_P(t) := \{x \mid at - \ell_P < x < at\}, \quad \ell_P := a\, G_1^{-1}\left(\frac{2a}{b}\right),$$

$$\varphi_P(z) := \begin{cases} G_1\left(\dfrac{(-z)^+}{a}\right), & \text{if } -\ell_P \le z, \\ G_0\left(-\dfrac{z + \ell_P}{a} + G_0^{-1}\left(\dfrac{2a}{b}\right)\right), & \text{if } z \le -\ell_P. \end{cases}$$

Here, we use the notation $(\xi)^+ := \max\{\xi, 0\}$. From Theorem 2.1 and the profile of G_i $(i = 0, 1)$, we see that the wave profile $\varphi_F(\cdot)$ satisfies

$$\varphi_F(z) = 0 \text{ for } z \ge 0, \quad \varphi_F'(z) < 0 \text{ for } z < 0$$

and the wave profile $\varphi_P(\cdot)$ satisfies

$$\begin{aligned} &\varphi_P(z) = 0 \text{ for } z \ge 0, \\ &\varphi_P'(z) < 0 \text{ for } -\ell_P < z < 0 \text{ with } \varphi_P(-\ell_P) = 2a/b, \\ &\varphi_P'(z) > 0 \text{ for } z < -\ell_P \text{ with } \varphi_P(-\infty) = 0. \end{aligned}$$

This is essentially shown in [18], but is not explicitly stated. For the readers' convenience, the proof of Theorem 2.1 is given in Section 3.2. Moreover, the traveling front (resp. pulse) solution is unique up to translation (see Remark 3.7).

To study the global dynamics of (1.5), we consider the following initial value problem:

$$\begin{cases} V = W(v), & x \in \partial\Omega(t),\ t > 0, \\ v_t = g(\mathbf{1}_{\Omega(t)}, v), & x \in \mathbb{R},\ t > 0, \\ \Omega(0) = \Omega_0 := \bigcup_{j=1}^{m}(x^0_{2j-1}, x^0_{2j}), \quad v(x,0) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.3}$$

where the initial data satisfies

(H1) (Boundedness) Ω_0 consists of m disjoint bounded intervals and $v_0 \geq 0$ is a bounded Lipschitz function defined in $\mathbb{R}$ with

$$M := \|v_0\|_{L^\infty(\mathbb{R})}. \tag{2.4}$$

(H2) (Well-posedness) $W(v_0(x)) \neq 0$ for any $x \in \partial\Omega(0)$.

Note that condition **(H2)** is demanded to guarantee the well-posedness of (2.3). We refer to [9] for a more detailed explanation. Let us recall the definition of a classical solution from [9] as follows.

Definition 2.2. (i) (Ω, v) is called a classical solution of (2.3) for $0 \leq t \leq T$ if there exist $x_k \in C^1([0,T])$, $k = 1, ..., 2m$, and

$$v \in C(\mathbb{R} \times [0,T]) \cap C^1\Big(\mathbb{R} \times (0,T] \setminus \{x = x_k(t),\ t \in [0,T],\ k = 1, ..., 2m\}\Big)$$

such that

$$x_1(t) < \cdots < x_{2j-1}(t) < x_{2j}(t) < x_{2j+1}(t) < \cdots < x_{2m}(t), \quad 0 \leq t \leq T,$$
$$\Omega := \bigcup_{0\leq t\leq T} [\Omega(t) \times \{t\}] = \bigcup_{0\leq t\leq T} \left[\bigcup_{j=1}^{m} (x_{2j-1}(t), x_{2j}(t)) \times \{t\}\right]$$

and the following equations hold pointwisely:

$$x_k'(t) = (-1)^k W\Big(v(x_k(t), t)\Big) := (-1)^k\Big(a - bv(x_k(t), t)\Big),\ 0 \leq t \leq T,\ k = 1, ..., 2m, \tag{2.5}$$

$$v_t = g(\mathbf{1}_{\Omega(t)}, v) \quad \text{in } \mathbb{R} \times [0,T], \tag{2.6}$$

$$x_k(0) = x_k^0, \quad v(x,0) = v_0(x). \tag{2.7}$$

(ii) (Ω, v) is called a classical solution of (2.3) for $0 \leq t < T$ if it is a classical solution for $0 \leq t \leq \tau$ for all $0 < \tau < T$.
(iii) (Ω, v) is called a classical solution of (1.5) for $\tau \leq t \leq T$ for some $\tau > 0$ if (i) holds with $t = 0$ replaced by $t = \tau$.
(iv) (Ω, v) is also called a non-negative classical solution of (1.5) for $\tau \leq t \leq T$ for some $\tau > 0$ if (iii) holds and $v \geq 0$.

Next, we recall the notion of the arrival time (cf.[18]) as follows, which play a crucial role in our analysis.

Definition 2.3. Let (Ω, v) be a classical solution and $x = x_k(\cdot)$ be strictly monotone in t for each k. Given any $y \in \mathbb{R}$, we say that $T_k(y)$ is the arrival time of the interface $x = x_k(t)$ for some $k \in \{1, ..., 2m\}$ to y if $y = x_k(T_k(y))$. For convenience, we define $T_k(y) := 0$ if $y \leq x_k(0)$ with $x_k'(0) > 0$ or $y \geq x_k(0)$ with $x_k'(0) < 0$.

We remark that the arrival time $t = T_k(y)$ can be almost seen as the inverse function of $t = x_k^{-1}(y)$. However, we allow the arrival time to be defined as 0 in some y.

Note that two faced interfaces may collide with each other (called an *annihilation*) at some time, and so the classical solution cannot be extended beyond this time. This leads us to introduce the annihilation time as

$$T_A = T_A(\Omega_0, v_0) := \sup\{\tau > 0 |\, x_k(t) < x_{k+1}(t) \text{ for all } t \in [0,\tau) \text{ and } k = 1, ..., 2m-1\}, \tag{2.8}$$

where m is the number of intervals of $\Omega(0)$. Note that $0 < T_A \leq \infty$ because of **(H2)**. If $T_A = \infty$, then the classical solution exists globally in time. If $T_A < \infty$, then there are two adjacent interfaces colliding with each other at time $t = T_A$, in which case we need to introduce the weak solutions to consider the continuation of the solution beyond the annihilation time.

Hereafter, for convenience, we shall use the following notations:

$$Q_T := \mathbb{R} \times [0,T], \quad \Omega^c := Q_T \backslash \Omega,$$

where $\Omega \subset \mathbb{R}^2$.

We now recall the definition of weak solutions from [9]. Let

$$X_T := \left\{ (\Omega, v) \;\middle|\; \begin{array}{l} v \in C(Q_T) \text{ and is Lipschitz continuous in } x, \\ \Omega \subset Q_T,\ \partial\Omega \text{ is Lipschitz}, \\ v \neq a/b \text{ on } \overline{\bigcup_{0\leq t\leq T} \partial\Omega(t) \times \{t\}}\,, \\ v(x,0) = v_0(x),\ \Omega(0) = \Omega_0 := \bigcup_{j=1}^m (x_{2j-1}^0, x_{2j}^0). \end{array} \right\},$$

where

$$\Omega(t) := \mathrm{int}_{\mathbb{R}}\{x \in \mathbb{R} \mid (x,t) \in \overline{\Omega}\}.$$

Remark that $\overline{\bigcup_{0\leq t\leq T} \partial\Omega(t) \times \{t\}}$ represents to the set of all interfaces in $[0,T]$ when it is a classical solution. Since $\partial\Omega$ is Lipschitz, the unit outer normal vector to $\partial\Omega$ exists almost everywhere, which is denoted by $\mathbf{n} := (n_1, n_2)$.

Definition 2.4. (i) We say that a pair (Ω, v) is a weak solution of (2.3) for $0 \leq t \leq T$ if $(\Omega, v) \in X_T$ and satisfies the following conditions **(C1)** and **(C2)**.

(C1) For any $\varphi \in H^1((0,T); L^2(\mathbb{R}))$ and $\psi \in H^1((0,T); L^2(\mathbb{R}))$ such that ψ has a compact support in $\mathbb{R} \times [0,T]$,

$$\left[\int_{\mathbb{R}} \mathbf{1}_{\Omega(t)} \varphi dx\right]_0^T = \int_0^T \int_{\Omega(t)} \varphi_t dx dt + \int_{\partial\Omega \cap (\mathbb{R}\times(0,T))} W(v)\varphi |n_1| d\sigma, \tag{2.9}$$

$$\left[\int_{\mathbb{R}} v\psi dx\right]_0^T = \int_0^T \int_{\mathbb{R}} \Big(v\psi_t + g(\mathbf{1}_{\Omega(t)}, v)\psi\Big) dx dt. \tag{2.10}$$

(C2) If $B(x_0, r_0) \times \{t_0\} \subset \Omega$ (resp. $\subset \Omega^c$) for some $r_0 > 0$ and $t_0 \in [0,T)$, then there exists $\tau_0 \in (0, T - t_0]$ depending only on r_0 such that

$$\{x_0\} \times [t_0, t_0 + \tau_0] \subset \Omega \text{ (resp. } \subset \Omega^c).$$

(ii) We say that a pair (Ω, v) is a weak solution of (2.3) for $0 \leq t < T$ if it is a weak solution for $0 \leq t \leq \tau$ for all $\tau \in (0,T)$.

Note that Definition 2.2 excludes the possibility of the generation of new interfaces. Indeed, for weak solutions, new interfaces may generate from any positive time such that the uniqueness of solutions does not hold in general. To prevent the nucleation of interfaces, we need an extra condition **(C2)** similar to condition (iii) of [6]. See [9] for more detailed explanation.

With the notion of classical and weak solutions, the following three results are established in [9].

Proposition A. ([9, Theorem 2.4]) *Assume* **(H1)** *and* **(H2)**. *Then problem* (2.3) *has a unique local in time non-negative classical solution.*

Proposition B. ([9, Proposition 3.8]) *Assume* **(H1)** *and* **(H2)**. *Then the classical solution of problem* (2.3) *can be extended uniquely until an annihilation occurs. Moreover,* $x_k(t)$ *is strictly monotone in* $[0, T_A)$, *where* T_A *is defined in* (2.8). *If* $T_A < \infty$, *there exists a positive constant* δ *such that* $|x_k'(t)| \geq \delta$ *for all* $t \in [0, T_A)$ *and* $k = 1, ..., 2m$.

Proposition C. ([9, Theorem 2.5]) *Assume* **(H1)** *and* **(H2)**. *Then there is a unique global in time weak solution to problem* (2.3).

Remark 2.5. It has been shown in [9] that if (Ω, v) is a weak solution for $t \in [t_0, t_1]$, then there exists an integer N such that (Ω, v) becomes a classical solution for $t \in [\tau_i, \tau_{i+1})$ for some $i = 0, 1, ..., N$ with $\tau_0 = t_0$ and $\tau_N = t_1$, where τ_i is an annihilation time for $1 \leq i \leq N-1$.

We now state the main results of this work as follows.

Theorem 2.6. *Assume* **(H1)** *and* **(H2)**. *Then any global weak solution of* (2.3) *must be one of the following types:*

(I) *there exists* $T > 0$ *such that* $\Omega(t) = \emptyset$ *for all* $t \geq T$ *and* $\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |v(x,t)| = 0$.

(II) *there exist* $x_{-2m_-} < \cdots < x_{-1} < x_1 < \cdots < x_{2m_+}$ *for some non-negative integers* $m_\pm$ *satisfying* $m \geq m_+ + m_- \neq 0$ *such that the following hold:*

$$\lim_{t\to\infty} x_k'(t) = \pm a, \quad k = \pm 1, \cdots, \pm 2m_\pm, \tag{2.11}$$

$$\lim_{t\to\infty} (x_{-2j+1}(t) - x_{-2j}(t)) = \ell_P, \quad j = 1, 2, \cdots, m_-, \tag{2.12}$$

$$\lim_{t\to\infty} (x_{-2j+2}(t) - x_{-2j+1}(t)) = \infty, \quad j = 2, ..., m_-, \tag{2.13}$$

$$\lim_{t\to\infty} (x_{2j-1}(t) - x_{2j-2}(t)) = \infty, \quad j = 2, ..., m_+, \tag{2.14}$$

$$\lim_{t\to\infty} (x_{2j}(t) - x_{2j-1}(t)) = \ell_P, \quad j = 1, \cdots, m_+, \tag{2.15}$$

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |v(x,t) - v_\infty(x,t)| = 0, \tag{2.16}$$

where ℓ_P *is defined in Theorem 2.1 and*

$$v_\infty(x,t) := \sum_{j=-m_-}^{-1} \varphi_P(x_{2j}(t) - x) + \sum_{j=1}^{m_+} \varphi_P(x - x_{2j}(t)); \tag{2.17}$$

(III) *there are* $x_{-2m_-} < \cdots < x_{-1} < y_{-1} < y_1 < x_1 < \cdots < x_{2m_+}$ *satisfying* (2.11)–(2.15), $m \geq m_+ + m_- + 1$ *such that the following hold:*

$$\lim_{t\to\infty} |y_{\pm 1}(t) - x_{\pm 1}(t)| = \infty \quad \text{if } m_\pm \geq 1, \tag{2.18}$$

$$\lim_{t\to\infty} \sup_{x\in I_K(t)} |v(x,t) - v_\infty(x,t)| = 0 \quad \text{for any positive constant } K, \tag{2.19}$$

where

$$I_K(t) := (-\infty, y_{-1}(t) + K] \cup [y_1(t) - K, \infty),$$

$$v_\infty(x,t) := \sum_{j=-m_-}^{-1} \varphi_P(x_{2j}(t) - x) + \Lambda(x; y_{-1}(t), y_1(t)) + \sum_{j=1}^{m_+} \varphi_P(x - x_{2j}(t)), \tag{2.20}$$

$$\Lambda(x; y_{-1}, y_1) := \varphi_F\left(\left|\frac{y_{-1}+y_1}{2} - x\right| - \frac{y_1 - y_{-1}}{2}\right). \tag{2.21}$$

For convenience, we say that the solution is of type (I) (resp. (II), (III)) if it satisfies (I) (resp. (II), (III)) of Theorem 2.6.

Remark 2.7. Conclusion (I) of Theorem 2.6 means that all interfaces collide during a finite time interval and no interface exists eventually. Hence, all excitation intervals disappear and the whole space becomes the resting state. For (II), the system eventually has $m_+ + m_-$ disjoint excitation intervals. If $m_- = 0$ (resp. $m_+ = 0$), then it is regarded as the existence of $x_1 < \cdots < x_{m_+}$ (resp. $x_{-m_-} < \cdots < x_{-1}$) only. This means that the solution converges to a series of traveling pulses propagating in the same direction with the speed a. Note that $m_+ + m_- \neq 0$. Otherwise, this case reduces to (I). For (III), which is different from (II), $x = y_{\pm 1}(t)$ is an interface that describes a traveling front. A solution of type (III) means that the solution converges to a wave that contains two traveling fronts following to a series of traveling pulses propagating in different directions.

We simply consider the case $m = 1$ to illustrate that each case in Theorem 2.6 can occur.

Theorem 2.8. *Assume* **(H1)**–**(H2)** *and let* (Ω, v) *be a global weak solution of* (2.3), *where* $\Omega(t) = (x_1(t), x_2(t))$ *with* $\Omega(0) = (x_1^0, x_2^0)$ *with* $-\infty < x_1^0 < x_2^0 < \infty$. *Then, the following hold*:

(i) *If* $W(v_0(x_1^0)) < 0$ *and* $W(v_0(x_2^0)) < 0$, *then* $T_A < \infty$, $\Omega(t) = \emptyset$ *for* $t > T_A$ *and*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |v(x,t)| = 0.$$

In this case, the solution is of type (I).

(ii) *If* $W(v_0(x_1^0)) < 0$ *and* $W(v_0(x_2^0)) > 0$, *then* $T_A = \infty$ *and*

$$\lim_{t\to\infty}(x_i(t) - at) = x_{i,\infty}, \quad \lim_{t\to\infty} x_i'(t) = a, \ i = 1, 2,$$
$$\lim_{t\to\infty}(x_2(t) - x_1(t)) = \ell_P, \quad \lim_{t\to\infty}\sup_{x\in\mathbb{R}} |v(x,t) - \varphi_P(x - x_2(t))| = 0$$

for some constants $x_{i,\infty}$ $(i = 1, 2)$. *In this case, the solution is of type* (II).

(iii) *If* $W(v_0(x_1^0)) > 0$ *and* $W(v_0(x_2^0)) < 0$, *then* $T_A = \infty$ *and*

$$\lim_{t\to\infty}(x_i(t) + at) = x_{i,\infty}, \quad \lim_{t\to\infty} x_i'(t) = -a, \ i = 1, 2,$$
$$\lim_{t\to\infty}(x_2(t) - x_1(t)) = \ell_P, \quad \lim_{t\to\infty}\sup_{x\in\mathbb{R}} |v(x,t) - \varphi_P(x_1(t) - x)| = 0$$

for some constants $x_{i,\infty}$ $(i = 1, 2)$. *In this case, the solution is of type* (II).

(iv) *If* $W(v_0(x_1^0)) > 0$ *and* $W(v_0(x_2^0)) > 0$, *then* $T_A = \infty$ *and*

$$\lim_{t\to\infty}[x_i(t) - (-1)^i at] = x_{i,\infty}, \quad \lim_{t\to\infty} x_i'(t) = (-1)^i a, \ i = 1, 2,$$
$$\lim_{t\to\infty}(x_2(t) - x_1(t)) = \infty, \quad \lim_{t\to\infty}\sup_{x\in I_K(t)} |v(x,t) - \Lambda(x; x_1(t), x_2(t))| = 0,$$

where K is any positive number,

$$I_K(t) := (-\infty, x_1(t) + K] \cup [x_2(t) - K, \infty)$$

and Λ is defined in (2.21). *In this case, the solution is of type* (III).

3. The global dynamics

We divide this section into three subsections. In the first subsection, we provide some useful a priori estimates. In Subsection 3.2, we shall prove Theorem 2.1. Besides, we consider the global dynamics of problem (2.3) when the number of the initial excitation interval is one, i.e., $m = 1$, and show Theorem 2.8. In Subsection 3.3, the global dynamics in the general case ($m > 1$) will be discussed and Theorem 2.6 will be proved.

3.1. **Some a priori estimates.**

The first lemma provides some useful estimates about the right-most interface $x = x_{2m}(t)$.

Lemma 3.1. *Assume that (Ω, v) is a non-negative classical solution of* (1.5) *for $t \in [t_0, T)$ with $x'_{2m}(\cdot) > 0$. Then, for $t_0 \le t < T$, the following hold:*

(i) $0 \le v(x,t) \le v(x,t_0)e^{-\gamma(t_0)(t-t_0)}$ *for* $x \ge x_{2m}(t)$,

(ii) $a(t-t_0) - \dfrac{b \sup_{z\in[x_{2m}(t_0),x_{2m}(t)]} v(z,t_0)}{\gamma(t_0)}\left[1 - e^{-\gamma(t_0)(t-t_0)}\right] \le x_{2m}(t) - x_{2m}(t_0) \le a(t-t_0)$,

(iii) $\left(a - bMe^{-\gamma(0)t_1}\right)(t - t_1) \le x_{2m}(t) - x_{2m}(t_1) \le a(t - t_1)$ *if* $t_0 = 0$, $0 < t_1 < t$,

(iv) $\lim_{t\to\infty}(x_{2m}(t) - at) = x_\infty$ *for some* $x_\infty \in \mathbb{R}$ *and* $\lim_{t\to\infty} x'_{2m}(t) = a$ *if* $T = \infty$,

where M is given in (2.4) *and*

$$\gamma(t) := \frac{g_2}{g_3\|v(\cdot,t)\|_{L^\infty(\mathbb{R})} + g_4} > 0. \tag{3.1}$$

Proof. By (2.6), we see that v is decreasing in t for $x \ge x_{2m}(t)$ and $t \in [t_0, T)$. Hence,

$$v_t = g(0,v) = -\frac{g_2 v}{g_3 v(x,t) + g_4} \le -\frac{g_2 v}{g_3 v(x,t_0) + g_4} \quad \text{for } x \ge x_{2m}(t) \text{ and } t \in [t_0, T).$$

Integrating over $[t_0, t]$ yields (i).

Recall from (2.5) that $x'_{2m}(t) = a - bv(x_{2m}(t), t)$ for $t \in [t_0, T)$, which together with (i) imply

$$a - bv(x_{2m}(t), t_0)e^{-\gamma(t_0)(t-t_0)} \le x'_{2m}(t) \le a. \tag{3.2}$$

By integrating (3.2) over $[t_0, t]$, we obtain (ii). Next, taking $t_0 = 0$ into (3.2), we have

$$a - bMe^{-\gamma(0)t} \le x'_{2m}(t) \le a. \tag{3.3}$$

By integrating (3.3) over $[t_1, t]$, we obtain (iii) immediately. For (iv), it follows from (3.2) that $x'_{2m}(t) \to a$ as $t \to \infty$ and $\sup_{t\ge 0} |x_{2m}(t) - at| < \infty$. Also, since $x_{2m}(t) - at$ is monotone decreasing in t, $x_{2m}(t) - at$ converges to some x_∞ and then (iv) holds. This completes the proof. □

By using the similar argument as in the proof of Lemma 3.1, we also have the following results for the left-most interface $x = x_1(t)$.

Lemma 3.2. *Assume that (Ω, v) is a non-negative classical solution of* (1.5) *for $t \in [t_0, T)$ with $x'_1(\cdot) < 0$. Then, for $t_0 \le t < T$, the following hold:*

(i) $0 \le v(x,t) \le v(x,t_0)e^{-\gamma(t_0)(t-t_0)}$ *for* $x \le x_1(t)$,

(ii) $a(t-t_0) - \dfrac{b\sup_{z\in[x_1(t),x_1(t_0)]} v(z,t_0)}{\gamma(t_0)}\left[1-e^{-\gamma(t_0)(t-t_0)}\right] \le x_1(t_0)-x_1(t) \le a(t-t_0)$,

(iii) $\left(a - bMe^{-\gamma(0)t_1}\right)(t-t_1) \le x_1(t_1)-x_1(t) \le a(t-t_1)$ *if* $t_0 = 0$, $0 < t_1 < t$,

(iv) $\lim_{t\to\infty}(x_1(t)+at) = x_\infty$ *for some* $x_\infty \in \mathbb{R}$ *and* $\lim_{t\to\infty} x_1'(t) = -a$ *if* $T = \infty$,

where M *and* $\gamma(t)$ *are given in* (2.4) *and* (3.1), *respectively.*

Next, we consider the motions of inner interfaces.

Lemma 3.3. *Assume that* (Ω, v) *is a non-negative classical solution of* (1.5) *for* $t \in [t_0,\infty)$ *with* $x_k'(\cdot) > 0$ *for* $k = 2j-1,\cdots,2m$ *with some* j. *Then there exists a positive constant* C *such that*

$$0 < x_{2j}(t) - x_{2j-1}(t) \le C \quad \text{for all } t \ge t_0.$$

Proof. Set $\ell(t) := x_{2j}(t) - x_{2j-1}(t)$. By the definition of the classical solution, we see that $\ell(t) > 0$ for all $t \ge 0$. It suffices to show the existence of an upper bound C. For contradiction, we assume that there exists a sequence $\{t_n\}$ such that

$$t_n \uparrow \infty \text{ and } \ell(t_n) \to \infty \text{ as } n \to \infty \text{ and } \ell'(t_n) \ge 0 \text{ for each } n. \tag{3.4}$$

Now we can take any $t_n \gg 1$ such that $H(t_n) := t_n - T_{2j}(x_{2j-1}(t_n)) > 0$, where $T_{2j}(z)$ is the arrival time of $x = x_{2j}(t)$ to z. Then, by the mean value theorem and (2.5), we have

$$\frac{\ell(t_n)}{H(t_n)} = x_{2j}'(\xi_n) = a - bv \le a.$$

where $\xi_n \in (T_{2j}(x_{2j-1}(t_n)), t_n)$. Since $\ell(t_n) \to \infty$, we must have $H(t_n) \to \infty$ as $n \to \infty$. However, note that

$$v_t(x_{2j-1}(t_n), t) = g(1, v(x_{2j-1}(t_n), t)), \quad t \in (T_{2j}(x_{2j-1}(t_n)), t_n).$$

By integrating it over $(T_{2j}(x_{2j-1}(t_n)), t_n)$, we have

$$\begin{aligned} v(x_{2j-1}(t_n), t_n) &= v(x_{2j-1}(t_n), T_{2j}(x_{2j-1}(t_n))) + \int_{T_{2j}(x_{2j-1}(t_n))}^{t_n} g(1,s)ds \\ &\ge \left(g_1 - \frac{g_2}{g_3}\right)H(t_n) \to \infty \quad \text{as } n \to \infty. \end{aligned}$$

By (2.5), we have $\ell'(t_n) = 2a - bv(x_{2j-1}(t_n), t_n) - bv(x_{2j}(t_n), t) \to -\infty$ as $n \to \infty$, which contradicts with (3.4). This completes the proof. □

3.2. **One excitation interval.** We first give a proof of Theorem 2.1.

Proof of Theorem 2.1. We show the existence of a traveling pulse (Ω, φ). Using the moving coordinate $z = x - ct$, from (1.5) we have

$$-c\varphi_z / g(0,\varphi) = 1 \qquad \text{for } z \ge 0 \text{ or } z < -\ell_P, \tag{3.5}$$

$$-c\varphi_z / g(1,\varphi) = 1 \qquad \text{for } -\ell_P < z < 0, \tag{3.6}$$

if $\Omega(t) = \{z + ct \mid -\ell_P < z < 0\}$ for some $\ell_P > 0$.

Since φ is bounded for $z > 0$, we must have $\varphi \equiv 0$ for $z \geq 0$. Indeed, for each $z_0 > 0$ by integrating (3.5) over (z_0, z), we have

$$-\int_{\varphi(z_0)}^{\varphi(z)} \frac{ds}{g(0,s)} = \frac{1}{c}(z - z_0).$$

Then $\varphi(z) \to \infty$ as $z \to \infty$ when $\varphi(z_0) > 0$. Hence the boundedness of φ implies that $\varphi \equiv 0$ for $z > 0$. By the continuity of the solution, $\varphi \equiv 0$ for $z \geq 0$. Thus, by integrating (3.6), we have $\varphi(z) = G_1((-z)^+/a)$ for $-\ell_P < z < 0$. On the other hand, since both speeds of the front and the back are c, we have

$$a - b\varphi(0) = c = -a + b\varphi(-\ell_P).$$

which implies that $c = a$ and that $\varphi(-\ell_P) = 2a/b$. Thus, by integrating (3.5) over $(z, -\ell_P)$,

$$\int_{\varphi(z)}^{2a/b} \frac{ds}{g(0,s)} = \frac{\ell_P + z}{a} \quad \text{for } z < -\ell_P.$$

Thus we see that

$$G_0^{-1}(2a/b) - G_0^{-1}(\varphi) = \frac{\ell_P + z}{a} \quad \text{for } z < -\ell_P.$$

That is,

$$\varphi(z) = G_0\left(-\frac{\ell_P + z}{a} + G_0^{-1}(2a/b)\right) \quad \text{for } z < -\ell_P.$$

This proves the existence of a traveling pulse.

Define

$$\varphi_F(z) := G_1\left(\frac{(-z)^+}{a}\right) \quad \text{for } z \in \mathbb{R}.$$

It is easy to see that $(\Omega_F(t), \varphi_F(x - at))$ is a traveling front solution. This completes the proof. ☐

To prove Theorem 2.8, we prepare the following three lemmas.

Lemma 3.4. *Assume* $\Omega_0 := (x_1^0, x_2^0)$. *Moreover, assume that*

$$W(v_0(x_1^0)) > 0, \quad W(v_0(x_2^0)) > 0.$$

Then, there are a unique strictly decreasing function $x_1(\cdot) \in C^1([0,\infty))$ *with* $x_1(0) = x_1^0$ *and a unique strictly increasing function* $x_2(\cdot) \in C^1([0,\infty))$ *with* $x_2(0) = x_2^0$ *such that*

$$\Omega(t) = \{x \in \mathbb{R} \mid x_1(t) < x < x_2(t)\}, \tag{3.7}$$

$$\lim_{t\to\infty} [x_i(t) - (-1)^i at] = x_{i,\infty}, \quad \lim_{t\to\infty} x_i'(t) = (-1)^i a, \ i = 1, 2, \tag{3.8}$$

$$\lim_{t\to\infty} (x_2(t) - x_1(t)) = \infty,$$

$$\lim_{t\to\infty} \sup_{x \in I_K(t)} |v(x,t) - \Lambda(x; x_1(t), x_2(t))| = 0, \tag{3.9}$$

where K *is any positive number,*

$$I_K(t) := (-\infty, x_1(t) + K] \cup [x_2(t) - K, \infty)$$

and Λ *is defined in* (2.21).

Proof. By Proposition B, there exists a unique strictly decreasing (resp. increasing) function $x = x_1(t) \in C^1([0,\infty))$ (resp. $x = x_2(t) \in C^1([0,\infty))$) with $x_1(0) = x_1^0$ (resp. $x_2(0) = x_2^0$) satisfying

$$x_1'(t) = -W(v(x_1(t),t)) := bv(x_1(t),t) - a, \quad t \geq 0$$
$$x_2'(t) = W(v(x_2(t),t)) := a - bv(x_2(t),t), \quad t \geq 0$$

such that (3.7) holds. Moreover, recall from [9] that $v \geq 0$. Note that $x = x_1(t)$ and $x = x_2(t)$ can be thought of as the left-most interface and the right-most interface, respectively. Hence, Lemmas 3.1- 3.2 are available to insure (3.8).

It suffices to show (3.9). Set

$$A_T := \{(x,t)|\, x_2(t) \leq x,\ t \geq T > 0\},$$
$$B_T := \{(x,t)|\, x_2(t) - K \leq x \leq x_2(t),\ t \geq T > 0\}.$$

By Lemma 3.1, for any given $\varepsilon > 0$, we can find $T_0 \gg 1$ such that

$$0 \leq v(x,t) < \varepsilon \quad \text{on } A_{T_0}, \tag{3.10}$$

$$|x_2(t) - at - x_\infty| < \varepsilon, \quad t \geq T_0. \tag{3.11}$$

Recall that $\varphi_F(x - x_2(t)) = 0$ on A_{T_0}. It follows from (3.10) that

$$\lim_{t\to\infty} \sup_{x\in[x_2(t),\infty)} |v(x,t) - \varphi_F(x - x_2(t))| = \lim_{t\to\infty} \sup_{x\in[x_2(t),\infty)} |v(x,t)| = 0. \tag{3.12}$$

Next, we focus on the region B_T. Recall from Definition 2.3 that $T_2(x)$ is the arrival time of the interface $x = x_2(t)$ at x. That is, $T_2(x_2(t)) = t$ for $t \geq 0$. From the equation of v, for $x_1^0 < x < x_2(t)$ and $t > T_2(x)$, we have

$$v(x,t) = G_1(G_1^{-1}(v(x,T_2(x))) + t - T_2(x)).$$

Also, Theorem 2.1 gives us

$$\varphi_F(x - x_2(t)) = G_1\Big(\frac{(x_2(t) - x)^+}{a}\Big) = G_1\Big(\frac{x_2(t) - x}{a}\Big), \quad \text{for } (x,t) \in B_{T_0}.$$

Hence, for $(x,t) \in B_{T_0}$, using the mean value theorem and (3.11), we have

$$\begin{aligned} |v(x,t) - \varphi_F(x - x_2(t))| &= \Big|G_1\Big(G_1^{-1}(v(x,T_2(x))) + t - T_2(x)\Big) - G_1\Big(\frac{x_2(t) - x}{a}\Big)\Big| \\ &\leq g_1\Big|G_1^{-1}(v(x,T_2(x))) + t - T_2(x) - \frac{x_2(t) - x}{a}\Big| \\ &\leq g_1\Big|G_1^{-1}(v(x,T_2(x)))\Big| + \frac{g_1}{a}|x - aT_2(x) - x_\infty| + \frac{g_1\varepsilon}{a}, \end{aligned} \tag{3.13}$$

where $\|G_1'\|_{L^\infty[0,\infty)} = g_1$.

On the other hand, by Lemma 3.1(ii) with $t_0 = T_0$ and $t = T_2(x)$, there exists a positive constant C such that

$$|x - aT_2(x) - x_\infty| \leq |x_2(T_0) - aT_0 - x_\infty| + C\Big(\sup_{z\in[x_2(T_0),x]} v(z,T_0)\Big)$$

for all $x \geq x_2(T_0)$. Because of (3.10), (3.11),

$$|x - aT_2(x) - x_\infty| \leq \varepsilon + C\varepsilon \quad \text{for all } x \geq x_2(T_0). \tag{3.14}$$

Taking $\tau \gg 1$ such that $(x,t) \in B_\tau$ implies that $x \geq x_2(T_0)$. Then (3.14) holds for all $(x,t) \in B_\tau$, and then the estimate (3.13) satisfies

$$|v(x,t) - \varphi_F(x - x_2(t))| \;\leq\; g_1\Big|G_1^{-1}(\varepsilon)\Big| + \frac{g_1}{a}(\varepsilon + C\varepsilon) + \frac{g_1\varepsilon}{a} \quad \text{for all } (x,t) \in B_\tau.$$

Together with the fact that $G_1^{-1}(\varepsilon) \to 0$ as $\varepsilon \to 0$, we see that

$$\lim_{t\to\infty} \sup_{x\in[x_2(t)-K,x_2(t)]} |v(x,t) - \varphi_F(x - x_2(t))| = 0. \tag{3.15}$$

Combining (3.12) and (3.15), we see that (3.9) holds for $[x_2(t) - K, \infty)$.

Similarly, the above process can apply (by the help of Lemma 3.2) to show

$$\lim_{t\to\infty} \sup_{x\in(-\infty,x_1(t)+K]} |v(x,t) - \varphi_F(x_1(t) - x)| = 0. \tag{3.16}$$

Also, since $x_1(t) \downarrow -\infty$ and $x_2(t) \uparrow \infty$ as $t \to \infty$, there exists a sufficiently large $T_K := T(K)$ such that

$$\Lambda(x; x_1(t), x_2(t)) := \begin{cases} \varphi_F(x - x_2(t)), & x \geq x_2(t) - K, \\ \varphi_F(x_1(t) - x), & x \leq x_1(t) + K \end{cases} \tag{3.17}$$

for all $t \geq T_K$. Combining (3.12), (3.15), (3.16) and (3.17), we see that (3.9) follows. This completes the proof. □

Lemma 3.5. *Assume that* $\Omega_0 := (x_1^0, x_2^0)$ *and*

$$W(v_0(x_1^0)) < 0, \quad W(v_0(x_2^0)) > 0.$$

Then there exists a unique strictly increasing function $x_i(\cdot) \in C^1([0,\infty))$ *with* $x_i(0) = x_i^0$ *for* $i = 1, 2$ *such that*

$$\Omega(t) = \{x \in \mathbb{R} \mid x_1(t) < x < x_2(t)\}, \tag{3.18}$$

$$\lim_{t\to\infty} [x_i(t) - at] = x_{i,\infty}, \quad \lim_{t\to\infty} x_i'(t) = a, \ i = 1, 2, \tag{3.19}$$

$$\lim_{t\to\infty} (x_2(t) - x_1(t)) = \ell_P, \tag{3.20}$$

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |v(x,t) - \varphi_P(x - x_2(t))| = 0. \tag{3.21}$$

Proof. By Proposition B, there exists a unique function $x_i(t)$ for $t \in [0, T_A)$ with $x_i(0) = x_i^0$ for $i = 1, 2$ such that (3.18) holds. In fact, $T_A = \infty$. For contradiction, we assume that $T_A < \infty$. Define $\ell(t) := x_2(t) - x_1(t)$. Then $\ell(T_A^-) = 0$. By (3.18) and Proposition B, there exists $\delta > 0$ such that

$$x_2'(t) = a - bv(x_2(t), t) \geq \delta, \quad t \in [0, T_A)$$
$$x_1'(t) = bv(x_2(t) - \ell(t), t) - a \geq \delta, \quad t \in [0, T_A).$$

Taking $t \to T_A^-$ and using $\ell(T_A^-) = 0$ yield

$$\frac{a+\delta}{b} \leq v(x_2(T_A^-), T_A^-) \leq \frac{a-\delta}{b},$$

which reaches a contradiction. Hence $T_A = \infty$ and then $\ell(t) > 0$ for $t \geq 0$. Next, by Lemma 3.1, we obtain (3.19) for $i = 2$.

Since $x_i' > 0$ $(i = 1, 2)$, the arrival time, denoted by $T_2(x)$ (resp. by $T_1(x)$), of the front (resp. the back) at a position x is well-defined. See Definition 2.3. Note that we have

$$x_2'(t) = a - bv(x_2(t), t), \quad \ell'(t) = F(\ell(t), x_2(t), t),$$

where

$$F(\ell, x_2, t) := 2a - bv(x_2 - \ell, t) - bv(x_2, t).$$

Then for any given small $\varepsilon > 0$, by Lemma 3.1 (iii) (with $m = 1$), there exists $t_* \gg 1$ such that

$$(a - b\varepsilon)(t - T_2(y)) \leq x_2(t) - y \leq a(t - T_2(y)) \tag{3.22}$$

for any y satisfying

$$y \le x_2(t) \quad \text{and} \quad T_2(y) \ge t_*. \tag{3.23}$$

By Lemma 3.3, $\ell(t)$ is bounded. Then,

$$x_1(t) = x_2(t) - \ell(t) \to \infty \quad \text{as } t \to \infty.$$

Thus, we can find $t_1 \gg 1$ such that

$$T_2(x_1(t)) \ge t_0 \quad \text{for all } t \ge t_1.$$

This implies that $y := x_1(t)$ satisfies (3.23). Hence we can put $y = x_1(t)$ into (3.22) such that

$$(a - b\varepsilon)\Big\{t - T_2(x_1(t))\Big\} \le \ell(t) \le a\Big\{t - T_2(x_1(t))\Big\} \tag{3.24}$$

for $t \ge t_1$.

On the other hand, from the definition of F, we easily obtain

$$2a - b\varepsilon - bG_1\Big(G_1^{-1}(\varepsilon) + t - T_2(x_1(t))\Big) \le \ell'(t) \le 2a - bG_1\Big(t - T_2(x_1(t))\Big)$$

for $t \ge t_1$. From (3.24), we obtain that

$$2a - b\varepsilon - bG_1\Big(G_1^{-1}(\varepsilon) + \frac{\ell(t)}{a - b\varepsilon}\Big) \le \ell'(t) = F(\ell(t), x_2(t), t) \le 2a - bG_1\Big(\frac{\ell(t)}{a}\Big) \tag{3.25}$$

for $t \ge t_1$. By the standard comparison principle for ODE, we have

$$(a - b\varepsilon)\left[G_1^{-1}\Big(\frac{2a}{b} - \varepsilon\Big) - G_1^{-1}(\varepsilon)\right] \le \liminf_{t\to\infty} \ell(t) \le \limsup_{t\to\infty} \ell(t) = aG_1^{-1}\Big(\frac{2a}{b}\Big).$$

Since $\varepsilon > 0$ can be arbitrary small, we obtain $\lim_{t\to\infty} \ell(t) = \ell_P$. Moreover, by (3.25), we see that $\lim_{t\to\infty} \ell'(t) = 0$. It follows that

$$\lim_{t\to\infty} x_1'(t) = \lim_{t\to\infty} (x_2'(t) - \ell'(t)) = a.$$

Hence, (3.19) for $i = 1$ and (3.20) follows.

We now deal with (3.21). Using G_0 and G_1, we can calculate the function v as follows:

(i) for $x < x_1(0)$,

$$v(x,t) = G_0\left(G_0^{-1}(v_0(x)) + t\right).$$

(ii) for $x_1(0) \le x < x_2(0)$,

$$v(x,t) = \begin{cases} G_1\left(G_1^{-1}(v_0(x)) + t\right), & \text{for } t \le T_1(x), \\ G_0\left(G_0^{-1}(v(x,T_1(x))) + t - T_1(x)\right), & \text{for } t > T_1(x). \end{cases}$$

(iii) for $x \ge x_2(0)$,

$$v(x,t) = \begin{cases} G_0\left(G_0^{-1}(v_0(x)) + t\right), & \text{for } t \le T_2(x), \\ G_1\left(G_1^{-1}(v(x,T_2(x))) + t - T_2(x)\right), & \text{for } T_2(x) < t \le T_1(x), \\ G_0\left(G_0^{-1}(v(x,T_1(x))) + t - T_1(x)\right), & \text{for } t > T_1(x). \end{cases}$$

By Lemma 3.4, (3.20) and the fact that $\varphi_P(z) = \varphi_F(z)$ for $z \ge -\ell_P$, we have that

$$\lim_{t\to\infty} \sup_{-\ell(t)\le z} |v(z + x_2(t), t) - \varphi_P(z)| = 0.$$

In particular, putting $z = -\ell(t)$ into the above limit, we have

$$\lim_{t\to\infty} \left| v(x_1(t), t) - \frac{2a}{b} \right| = 0,$$

which implies that for sufficiently small $\varepsilon > 0$, there is a positive constant t_2 satisfying

$$a - \varepsilon \le x_1'(t) \le a + \varepsilon \qquad \text{for } t \ge t_2.$$

Integrating over $[T_1(x), t]$ yields

$$(a - \varepsilon)(t - T_1(x)) \le x_1(t) - x \le (a + \varepsilon)(t - T_1(x))$$

for $t_2 \le T_1(x) \le t$. Namely, for $x \ge x_1(t_2)$ and $t \ge T_1(x)$, we have

$$\left| t - T_1(x) + \frac{x - x_1(t)}{a} \right| \le \frac{\varepsilon}{a} |t - T_1(x)|. \tag{3.26}$$

Since $t > T_1(x)$ where $x := z + x_2(t) < x_2(t) - \ell(t)$, we have

$$\begin{aligned}
&|v(x,t) - \varphi_P(x - x_2(t))| \\
&\le \left| G_0\left(G_0^{-1}(v(x, T_1(x))) + t - T_1(x)\right) - G_0\left(-\frac{x - x_1(t)}{a} + G_0^{-1}(v(x_1(t), t))\right) \right| \\
&\quad + \left| G_0\left(-\frac{x - x_1(t)}{a} + G_0^{-1}(v(x_1(t), t))\right) - G_0\left(-\frac{x - x_2(t) + \ell_P}{a} + G_0^{-1}\Big(\frac{2a}{b}\Big)\right) \right| \\
&\le \frac{g_2}{g_3} \left| t - T_1(x) + \frac{x - x_1(t)}{a} + G_0^{-1}(v(x, T_1(x))) - G_0^{-1}(v(x_1(t), t)) \right| \\
&\quad + \frac{g_2}{g_3} \left| \frac{\ell_P - \ell(t)}{a} + G_0^{-1}(v(x_1(t), t)) - G_0^{-1}\Big(\frac{2a}{b}\Big) \right|.
\end{aligned}$$

Thus, using (3.26) and the fact that $v(x, T_1(x)) \to 2a/b$ as $x \to \infty$, we have

$$\lim_{t\to\infty} \sup_{-K-\ell(t) \le z \le -\ell(t)} |v(z + x_2(t), t) - \varphi_P(z)| = 0$$

for any positive constant K. Using $\lim_{s\to\infty} G_0(s) = 0$ and taking K sufficiently large, we get

$$\begin{aligned}
&\lim_{t\to\infty} \sup_{z \le -K-\ell(t)} |v(z + x_2(t), t) - \varphi_P(z)| \\
&\le \lim_{t\to\infty} \sup_{z \le -K-\ell(t)} |v(z + x_2(t), t)| + \lim_{t\to\infty} \sup_{z \le -K-\ell(t)} |\varphi_P(z)| = 0.
\end{aligned}$$

This completes the proof. □

By the same argument used in Lemma 3.5, we have the following result:

Lemma 3.6. *Assume that* $\Omega_0 := (x_1^0, x_2^0)$ *and*

$$W(v_0(x_1^0)) > 0, \quad W(v_0(x_2^0)) < 0.$$

Then there exists a unique strictly decreasing function $x_i(\cdot) \in C^1([0, \infty))$ *with* $x_i(0) = x_i^0$ *for* $i = 1, 2$ *such that*

$$\begin{aligned}
&\Omega(t) = \{x \in \mathbb{R} \mid x_1(t) < x < x_2(t)\}, \\
&\lim_{t\to\infty} [x_i(t) + at] = x_{i,\infty}, \quad \lim_{t\to\infty} x_i'(t) = -a, \ i = 1, 2, \\
&\lim_{t\to\infty} (x_2(t) - x_1(t)) = \ell_P, \\
&\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |v(x,t) - \varphi_P(x_1(t) - x)| = 0.
\end{aligned}$$

Thanks to Lemmas 3.5 and 3.6, we see that the traveling pulse solution is globally asymptotically stable.

Remark 3.7. From Lemmas 3.4–3.6, the traveling pulse (resp. front) solution is unique up to a translation.

We are ready to show Theorem 2.8.

Proof of Theorem 2.8. Let us first consider (i). Clearly, $\Omega(t) = \emptyset$ for all $t \geq T_A$, where $T_A < \infty$. Then the weak solution becomes a classical solution for $t \geq T_A$ (see [9]). By (2.6), we have $v_t = g(0, v)$ for $x \in \mathbb{R}$ and $t \geq T_A$ and, clearly, $\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |v(x,t)| = 0$. Note that (ii), (iii) and (iv) follow from Lemma 3.5, Lemma 3.6 and Lemma 3.4, respectively. This completes the proof. □

3.3. **Multiple excitation intervals.** In this subsection, we consider the dynamics of (2.3) in the general case where Ω_0 consists of m disjoint bounded intervals for some $m > 1$, and then Theorem 2.6 will be proved.

To characterize the global dynamics of (2.3), it is important to understand how the number of interfaces of (2.3) change in time. The symbolic dynamics is useful to determine the number of interfaces. We shall use the notations similar to those in [10]. Set

$$\mathcal{X} := \left\{ (\Omega, v) \;\middle|\; v \in C(\mathbb{R}),\ \Omega \text{ is a union of disjoint finite open intervals and } v|_{\partial\Omega} \neq \frac{a}{b} \right\},$$

where $\Omega = \cup_{j=1}^{m} (x_{2j-1}, x_{2j})$ and $x_1 \leq x_2 \leq \cdots \leq x_{2m}$.

For any given $(\Omega, v) \in \mathcal{X}$, we define $Z[(\Omega, v)]$ as the minimal number of disjoint intervals that compose $\overline{\Omega}$ (the closure of Ω). Clearly, $Z[(\Omega, v)] \leq m$. Moreover, $Z[(\Omega, v)] < m$ if $x_j = x_{j+1}$ for some j. When $\Omega = \emptyset$, we define $Z[(\Omega, v)] = 0$.

Next, denote $\mathrm{SGN}[(\Omega, v)]$ as the word consisting of letters `f` (i.e., front) and `b` (i.e., back) defined as follows: if $Z[(\Omega, v)] = N$ and $y_1 < y_2 < \cdots < y_{2N}$ with $y_i \in \partial\overline{\Omega}$, then

$$\mathrm{SGN}[(\Omega, v)] := [w_1 w_2 w_3 w_4 ... w_{2N-1} w_{2N}],$$

where

$$w_j = \begin{cases} \mathtt{f} & \text{if } W(v(y_j)) > 0, \\ \mathtt{b} & \text{if } W(v(y_j)) < 0. \end{cases}$$

Moreover, when $\Omega = \emptyset$, $\mathrm{SGN}[(\Omega, v)] := [\ ]$, i.e., the empty word. Clearly, the length of the word $\mathrm{SGN}[(\Omega, v)]$ is equal to $2Z[(\Omega, v)]$. We also denote the j-th letter w_j of $\mathrm{SGN}[(\Omega, v)]$ by

$$\mathrm{SGN}[(\Omega, v)]_{j,2N}$$

if $Z[(\Omega, v)] = N$ and $1 \leq j \leq 2N$.

We give some simple examples to illustrate the notions mentioned above.

Example 3.8. *Define $v(x) = a/b + x$ for $x \in \mathbb{R}$.*

(i) *Given $\Omega_1 = (-1, 1)$, we have $\overline{\Omega}_1 = [-1, 1]$ and $W(v(x)) = -bx$. It follows that*

$$Z[(\Omega_1, v)] = 1 \text{ and } \mathrm{SGN}[(\Omega_1, v)] = [\mathtt{fb}].$$

(ii) *Given $\Omega_2 = (-2, -1) \cup (1, 2)$, then $\overline{\Omega}_2 = [-2, -1] \cup [1, 2]$ and $W(v(x)) = -bx$. Hence, we have*

$$Z[(\Omega_2, v)] = 2 \text{ and } \mathrm{SGN}[(\Omega_2, v)] = [\mathtt{ffbb}].$$

Moreover, $\mathrm{SGN}[(\Omega_2, v)]_{3,4} = [\mathtt{ffbb}]_{3,4} = \mathtt{b}$.

(iii) *Given* $\Omega_3 = (-2,1) \cup (1,2)$*, then* $\overline{\Omega}_3 = [-2,2]$ *and* $W(v(x)) = -bx$*. Hence, we obtain*

$$Z[(\Omega_3, v)] = 1 \text{ and } \mathrm{SGN}[(\Omega_3, v)] = [\mathtt{fb}].$$

If A, B are two words consisting of f and b, we write $A \rhd B$ if B is a subword of A. Here a subword of $A = [w_1 w_2 w_3 ... w_{2N-1} w_{2N}]$ is one of $[w_{j_1} w_{j_2} \dots w_{j_{2M}}]$ where $0 \le M \le N$, $1 \le j_1 < j_2 < \cdots < j_{2M} \le 2N$.

For example, $[\mathtt{ffbb}] \rhd B$ for $B = [\mathtt{ffbb}], [\mathtt{ff}], [\mathtt{bb}], [\mathtt{fb}], [\,]$, but not for $B = [\mathtt{bf}]$.

From the definition of a subword, we can easily obtain

Lemma 3.9. *If* $A \rhd B \rhd C$*, then* $A \rhd C$*.*

We are ready to consider the global dynamics of solutions of (2.3) in terms of the notions given above. Let (Ω, v) be any global weak solution of (2.3) with $Z[\Omega_0, v_0] = m$. By the definition of weak solutions and Remark 2.5, we see that $(\Omega(t), v(\cdot, t)) \in \mathcal{X}$ for each $t \ge 0$. It follows that $Z[(\Omega(t), v(\cdot, t)]$ is well defined for any $t \ge 0$. In addition, by Remark 2.5 and Proposition B, we see that $W(v(x_k, t)) \ne 0$ for all $x_k \in \partial\Omega(t)$, which implies that $W(v(y_k, t)) \ne 0$ for all $y_k \in \partial\overline{\Omega}(t)$. It follows that $\mathrm{SGN}[(\Omega(t), v(\cdot, t)]$ is well defined for any $t \ge 0$.

We provide the following proposition for Z and SGN.

Proposition 3.10. *Let* (Ω, v) *be a global weak solution of* (2.3)*. Then, the following hold:*

(i) *If an annihilation of* $(\Omega(t), v(\cdot, t))$ *does not occur for* $t_1 \le t < t_2$*, then* $Z[(\Omega(t), v(\cdot, t))]$ *and* $\mathrm{SGN}[(\Omega(t), v(\cdot, t))]$ *does not change for* $t_1 \le t < t_2$*.*

(ii) *If an annihilation occurs at some time* $t_0 > 0$*, then* $Z[(\Omega(t), v(\cdot, t))]$ *drops* k *at* $t = t_0$ *and the length of* $\mathrm{SGN}[(\Omega(t), v(\cdot, t))]$ *drops* $2k$ *at* $t = t_0$ *for some* $k \in \mathbb{N}$*. Moreover,* $\mathrm{SGN}[(\Omega(t_0), v(\cdot, t_0))]$ *is a subword of* $\mathrm{SGN}[(\Omega(t_0 - \epsilon), v(\cdot, t_0 - \epsilon))]$ *for sufficiently small* $\epsilon > 0$*.*

(iii) $Z[(\Omega(t), v(\cdot, t))]$ *and* $\mathrm{SGN}[(\Omega(t), v(\cdot, t))]$ *are nonincreasing in* t*. Namely,*

$$Z[(\Omega(t), v(\cdot, t))] \ge Z[(\Omega(s), v(\cdot, s))] \quad \text{for any } s > t,$$
$$\mathrm{SGN}[(\Omega(t), v(\cdot, t))] \rhd \mathrm{SGN}[(\Omega(s), v(\cdot, s))] \quad \text{for any } s > t.$$

(iv) *Suppose that there exist* $\tau \ge 0$ *and* $j \in \{1, 2, \cdots, m\}$ *such that*

$$\mathrm{SGN}[(\Omega(\tau), v(\cdot, \tau))]_{2j,2N} = \mathrm{SGN}[(\Omega(\tau), v(\cdot, \tau))]_{2j+1,2N} = \mathtt{f}.$$

Then there exists $\tau' > \tau$ *such that* $x_{2j}(\tau') = x_{2j+1}(\tau')$*, where* $x_{2j}(t)$ *and* $x_{2j+1}(t)$ *are the positions of* $(2j)$*th and* $(2j+1)$*th fronts. Hence, an annihilation occurs at* $t = \tau'$*.*

(v) *Suppose that there exist* $\tau \ge 0$ *and* $j \in \{1, 2, \cdots, m\}$ *such that*

$$\mathrm{SGN}[(\Omega(\tau), v(\cdot, \tau))]_{2j-1,2N} = \mathrm{SGN}[(\Omega(\tau), v(\cdot, \tau))]_{2j,2N} = \mathtt{b}.$$

Then there exists $\tau' > 0$ *such that* $x_{2j-1}(\tau') = x_{2j}(\tau')$*, where* $x_{2j-1}(t)$ *and* $x_{2j}(t)$ *are the positions of* $(2j-1)$*th and* $(2j)$*th backs. Hence, an annihilation occurs at* $t = \tau'$*.*

(vi) *There exist* $T^* \gg 1$ *and* $\kappa \in \{0, 1, ..., m\}$ *such that* $Z[(\Omega(t), v(\cdot, t))] \equiv \kappa$ *for all* $t \ge T^*$ *and* $\mathrm{SGN}[(\Omega(t), v(\cdot, t))]$ *never changes for all* $t \ge T^*$ *with the length* 2κ*. Moreover,* (Ω, v) *becomes a classical solution of* (2.3) *for* $t \ge T^*$*.*

Proof. Recall Remark 2.5 that there exists $N \in \mathbb{N}$ such that (Ω, v) becomes a classical solution for $t \in [\tau_i, \tau_{i+1})$ for some $i = 0, 1, ..., N$ with $\tau_0 = 0$ and $\tau_{N+1} = \infty$, where τ_i is an annihilation time for $1 \le i \le N - 1$.

For (i), since no interface can generate (because of **(C2)**), the number of interfaces does not change for $t_1 \le t < t_2$. Hence, $Z[(\Omega(t), v(\cdot,t)]$ does not change for $t_1 \le t < t_2$. Moreover, by Proposition B, the absence of an annihilation implies that each interface $|x_k'(t)|$ never vanishes for $t_1 \le t < t_2$. Namely, $W(v(x_k(t),t))$ does not change sign for $t_1 \le t < t_2$. Hence, $\mathrm{SGN}[(\Omega(t), v(\cdot,t))]$ does not change for $t_1 \le t < t_2$.

Next we prove (ii). Note that an annihilation only occurs within two adjacent interfaces. By the definition of $Z[(\Omega(t), v(\cdot,t))]$, we see that any two adjacent interfaces collide each other at time t_0 implies that $Z[(\Omega(t), v(\cdot,t))]$ drops 1 at $t = t_0$. Suppose that there are k groups of two adjacent interfaces which collide at time t_0 for some $k \in \mathbb{N}$. Then $Z[(\Omega(t), v(\cdot,t))]$ drops k at $t = t_0$. Also, in this situation the boundary points of the closure of $\Omega(t_0)$ drop $2k$ and so the letters of $\mathrm{SGN}[(\Omega(t), v(\cdot,t))]$ drops $2k$ at $t = t_0$ and the dropped letters consist of k group of two adjacent letters. By the definition of subwords, $\mathrm{SGN}[(\Omega(t_0), v(\cdot,t_0))]$ is a subword of $\mathrm{SGN}[(\Omega(t_0-\epsilon), v(\cdot,t_0-\epsilon))]$ for sufficiently small $\epsilon > 0$.

For (iii), it follows from (i) and (ii) that $Z[(\Omega(t), v(\cdot,t))]$ is non-increasing in time. By (i), (ii) and Lemma 3.9, $\mathrm{SGN}[(\Omega(t), v(\cdot,t))]$ are nonincreasing in t.

We deal with (iv). By Proposition B, $x_{2j}'(t) \ge \delta$ and $x_{2j+1}'(t) \le -\delta$ for some $\delta > 0$. This implies the existence of τ'. Similarly, we have (v).

For (vi), we see from (iii) and the fact $0 \le Z[(\Omega(t), v(\cdot,t))] \le m$ that $Z[(\Omega(t), v(\cdot,t))] \to \kappa$ as $t \to \infty$ for some $\kappa \in \{0, 1, ..., m\}$. Since $Z[(\Omega(t), v(\cdot,t))] \in \mathbb{N} \cup \{0\}$, there exist $T^* \gg 1$ such that $Z[(\Omega(t), v(\cdot,t))] \equiv \kappa$ for all $t \ge T^*$. For $t > T^*$, v never takes a value a/b on any interface. It means that $\mathrm{SGN}[(\Omega(t), v(\cdot,t))]$ does not change for $t \ge T$. By the definition, the length of $\mathrm{SGN}[(\Omega(t), v(\cdot,t))]$ is 2κ. This completes the proof. ☐

We note that if $\mathrm{SGN}[(\Omega(0), v_0)]_{2j-1,2N} = \mathrm{SGN}[(\Omega(0), v_0)]_{2j,2N} = \mathtt{f}$, two fronts do not collide and the j-interval converges to $\Lambda(x; x_{2j-1}, x_{2j})$ as t tends to infinity if these fronts exist for all large time.

Given two words $A = [a_1 a_2 .... a_{2N}]$ and $B = [b_1 b_2 ... b_{2M}]$, we define

$$A \curlyvee B := [a_1 a_2 ... a_{2N} b_1 b_2 ... b_{2M}],$$
$$kA = \overbrace{A \curlyvee A \curlyvee ... \curlyvee A}^{k-\text{times}}, \quad k \in \mathbb{N}.$$

We also set $[\ ] \curlyvee A = A$, $A \curlyvee [\ ] = A$ and $0A = [\ ]$. Three different types of words are defined as follows:

the word of type (I): $[\ ]$,

the word of type (II): $m_-[\mathtt{fb}] \curlyvee m_+[\mathtt{bf}]$ where $m_\pm \in \mathbb{N} \cup \{0\}$ and $(m_-, m_+) \ne (0,0)$,

the word of type (III): $m_-[\mathtt{fb}] \curlyvee [\mathtt{ff}] \curlyvee m_+[\mathtt{bf}]$ where $m_\pm \in \mathbb{N} \cup \{0\}$.

Proposition 3.11. *Let* (Ω, v) *be a global weak solution of* (2.3). *Then, there exists* $T^* > 0$ *such that* $\mathrm{SGN}[(\Omega(T^*), v(\cdot,T^*))]$ *is one of the types* (I), (II) *or* (III).

Proof. By Proposition 3.10 (iv), there is a positive constant T^* such that $Z[(\Omega(t), v(\cdot,t))]$ and $\mathrm{SGN}[(\Omega(t), v(\cdot,t))]$ never change for all $t \ge T^*$. Set $N := Z[(\Omega(T^*), v(\cdot,T^*))] \le m$ and $\mathbf{w} := \mathrm{SGN}[(\Omega(t), v(\cdot,t))]$. If $N = 0$, $\mathbf{w} = [\ ]$, which belongs to the type (I). When $N = 1$, we divide into four cases:

(1) If $\mathbf{w} = [\mathtt{fb}]$, then $\mathbf{w}$ is of type (II);

(2) If $\mathbf{w} = [\texttt{ff}]$, then $\mathbf{w}$ is of type (III);
(3) If $\mathbf{w} = [\texttt{bb}]$, by Proposition 3.10(v), an annihilation occurs at some $t > T^*$. This contradicts the choice of T^*;
(4) If $\mathbf{w} = [\texttt{bf}]$, then $\mathbf{w}$ is of type (II).

From the above discussion, the conclusion holds for $N = 1$. Assume that there exists either $w_{2j-1}w_{2j} = \texttt{bb}$ with $j \in \{1, 2, \cdots, N\}$ or $w_{2j}w_{2j+1} = \texttt{ff}$ with $j \in \{1, 2, \cdots, N-1\}$. By Proposition 3.10, an annihilation occurs at some $t^* > T^*$, which means that $Z[(\Omega(t), v(\cdot, t))]$ drops at least 1 at $t = t^*$. This contradicts the choice of T^*. Now we claim that if $\mathbf{w}$ does not include $w_{2j-1}w_{2j} = \texttt{bb}$ nor $w_{2j}w_{2j+1} = \texttt{ff}$, then $\mathbf{w}$ is of type (II) or type (III) by the induction on $N \in \mathbb{N}$. The case of $N = 1$ has been shown already. Assume that this holds for $N = 1, \cdots, n$. Then, for $N = n+1$, $\mathbf{w} = [w_1w_2w_3 \cdots w_{2n+1}w_{2n+2}]$ with the length $2n+2$. The subword $\mathbf{w}' := [w_1w_2w_3 \cdots w_{2n-1}w_{2n}]$ does not include $w_{2j-1}w_{2j} = \texttt{bb}$ nor $w_{2j}w_{2j+1} = \texttt{ff}$. Hence $\mathbf{w}'$ is of type (II) or type (III). If $w_{2n} = \texttt{f}$, then w_{2n+1} must be $\texttt{b}$ because of the assumption. Then w_{2n+2} must be $\texttt{f}$. This means that $\mathbf{w} = \mathbf{w}' \curlyvee [\texttt{bf}]$ is of type (II) or type (III). Next consider the case where $w_{2n} = \texttt{b}$. We note that $\mathbf{w}'$ must be $n[\texttt{fb}]$ in this case. Then w_{2n+1} is arbitrary, namely, $\texttt{b}$ or $\texttt{f}$. For the former case, w_{2n+2} must be $\texttt{f}$. This means that $\mathbf{w} = \mathbf{w}' \curlyvee [\texttt{bf}]$ is of type (II). For the latter case, $w_{2n+1}w_{2n+2} = \texttt{fb}$ or $\texttt{ff}$. In the case of $w_{2n+1}w_{2n+2} = \texttt{fb}$, $\mathbf{w} = n[\texttt{fb}] \curlyvee [\texttt{fb}] = (n+1)[\texttt{fb}]$ is of type (II). In the case of $w_{2n+1}w_{2n+2} = \texttt{ff}$, $\mathbf{w} = n[\texttt{fb}] \curlyvee [\texttt{ff}]$ is of type (III). By the induction, this completes the proof of Proposition 3.11. □

Remark 3.12. From Proposition 3.11, we can investigate the possible global dynamics of (2.3) after $t = T^*$. By Propositions 3.10 (vi) and 3.11, $\mathrm{SGN}[(\Omega(T^*), v(\cdot, T^*))]$ is one of the types (I), (II) or (III) and we can assume that the solution is a classical solution for $t \geq T^*$. For the corresponding word $\mathrm{SGN}[(\Omega(T^*), v(\cdot, T^*))]$ being of type (I), the global dynamic is simple since $\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |v(x, t)| = 0$.

Hereafter, when $\mathrm{SGN}[(\Omega(T^*), v(\cdot, T^*))]$ is of type (II), we can write

$$\mathrm{SGN}[(\Omega(T^*), v(\cdot, T^*))] = m_-[\texttt{fb}] \curlyvee m_+[\texttt{bf}] \tag{3.27}$$

for some $m_\pm \in \mathbb{N} \cup \{0\}$ and $(m_-, m_+) \neq (0, 0)$.

When $\mathrm{SGN}[(\Omega(T^*), v(\cdot, T^*))]$ is of type (III), we can write

$$\mathrm{SGN}[(\Omega(T^*), v(\cdot, T^*))] = m_-[\texttt{fb}] \curlyvee [\texttt{ff}] \curlyvee m_+[\texttt{bf}] \tag{3.28}$$

for some $m_\pm \in \mathbb{N} \cup \{0\}$.

For the word being of type (II) or (III), we investigate the asymptotic distance between adjacent two interfaces as follows.

Proposition 3.13. *Let (Ω, v) be a classical solution for $t \geq T^*$ with interfaces*

$$x_{-2m_-}(t) < \cdots < x_{-1}(t) < x_1(t) < \cdots < x_{2m_+}(t)$$
$$(\textit{resp. } x_{-2m_-}(t) < \cdots < x_{-1}(t) < y_{-1}(t) < y_1(t) < x_1(t) < \cdots < x_{2m_+}(t))$$

and SGN$[(\Omega(T^*), v(\cdot, T^*))]$ *is of type* (II) (*resp. type* (III)). *Then as* $t \to \infty$,

$$x_{2j}(t) - x_{2j-1}(t) = \ell_P + O\Big(\frac{1}{t^{\alpha_j^+}}\Big) \quad \text{for } j = 1, ..., m_+, \tag{3.29}$$

$$x_{2j-1}(t) - x_{2j-2}(t) \ge \delta_j^+ \log t \quad \text{for } j = 2, ..., m_+, \tag{3.30}$$

$$x_{-2j+1}(t) - x_{-2j}(t) = \ell_P + O\Big(\frac{1}{t^{\alpha_j^-}}\Big) \quad \text{for } j = 1, ..., m_-, \tag{3.31}$$

$$x_{-2j+2}(t) - x_{-2j+1}(t) \ge \delta_j^- \log t \quad \text{for } j = 2, ..., m_-, \tag{3.32}$$

$$|x_{\pm 1}(t) - y_{\pm 1}(t)| \ge \delta_1^\pm \log t, \quad \lim_{t\to\infty} y'_\pm(t) = \pm a \quad \text{if } y_{\pm 1} \text{ exists}, \tag{3.33}$$

$$\lim_{t\to\infty} x'_k(t) = \pm a \quad \text{for } k = \pm 1, \cdots, \pm 2m_\pm, \tag{3.34}$$

where $\alpha_j^\pm$ *and* $\delta_j^\pm$ *are positive constants.*

To prove Proposition 3.13, we prepare the following lemmas.

Lemma 3.14. *Assume the same hypotheses of Proposition 3.13. Also, assume that there exist constants* $C > 0$ *and* $\alpha_k \in (0, 1)$ *such that*

$$0 \le v(x_{2k}(t), t) \le Ct^{-\alpha_k}, \quad t \ge T^* \tag{3.35}$$

for some $k \in \{1, 2, \cdots, m_+\}$. *Then*

$$x_{2k}(t) - x_{2k-1}(t) = \ell_P + O(t^{-\alpha_k}) \text{ as } t \to \infty.$$

Proof. Set $\ell(t) := x_{2k}(t) - x_{2k-1}(t)$. Then

$$\ell'(t) = x'_{2k}(t) - x'_{2k-1}(t) = 2a - bv(x_{2k}(t), t) - bv(x_{2k-1}(t), t). \tag{3.36}$$

By (3.35) and (2.5), we have

$$a - bCt^{-\alpha_k} \le x'_{2k}(t) \le a, \quad t \ge T^*. \tag{3.37}$$

Integrating (3.37) over (τ, t) with $\tau > T^*$ yields

$$\Big(a - \frac{bC}{\tau^{\alpha_k}}\Big)(t - \tau) \le x_{2k}(t) - x_{2k}(\tau) \le a(t - \tau).$$

Taking $t \gg 1$ such that $\tau = T_{2k}(x_{2k-1}(t)) > T^*$, where $T_{2k}(z)$ is the arrival time of $x = x_{2k}(t)$ to z, we have

$$\Big(a - \frac{bC}{T_{2k}(x_{2k-1}(t))^{\alpha_k}}\Big)(t - T_{2k}(x_{2k-1}(t))) \le \ell(t) \le a(t - T_{2k}(x_{2k-1}(t))). \tag{3.38}$$

In particular, from (3.38), we have

$$T_{2k}(x_{2k-1}(t)) = t - \frac{\ell(t)}{a + o(1)} \quad \text{as } t \to \infty.$$

Also, because of the boundedness of $\ell(\cdot)$ (Lemma 3.3), there exist $\kappa > 0$ and $T \ge T^*$ such that

$$T_{2k}(x_{2k-1}(t)) \ge t - \kappa \quad \text{for } t \ge T. \tag{3.39}$$

Together with (3.38), we see that

$$\frac{\ell(t)}{a} \le t - T_{2k}(x_{2k-1}(t)) \le \frac{\ell(t)}{a - bC(t-\kappa)^{-\alpha_k}}. \tag{3.40}$$

Taking T large enough, we can assume that

$$0 < \frac{a}{2} < a - bC(t - \kappa)^{-\alpha_k} \quad \text{for } t > T. \tag{3.41}$$

We now estimate $v(x_{2k-1}(t),t)$. For this, by (3.35) with t replaced by $T_{2k}(x_{2k-1}(t))$,

$$0 \le v\Big(x_{2k-1}(t), T_{2k}(x_{2k-1}(t))\Big) \le CT_{2k}(x_{2k-1}(t))^{-\alpha_k} \le C(t-\kappa)^{-\alpha_k}, \tag{3.42}$$

where the last inequality holds by using (3.39). Also, note that

$$v(x_{2k-1}(t),t) = G_1\Big(G_1^{-1}(v(x_{2k-1}(t), T_{2k}(x_{2k-1}(t)))) + t - T_{2k}(x_{2k-1}(t))\Big).$$

Then using (3.40), (3.42) and the strict monotonicity of G_1 and G_1^{-1} yield that

$$G_1\Big(\frac{\ell(t)}{a}\Big) \le v(x_{2k-1}(t),t) \le G_1\left(G_1^{-1}\Big(C(t-\kappa)^{-\alpha_k}\Big) + \frac{\ell(t)}{a-bC(t-\kappa)^{-\alpha_k}}\right). \tag{3.43}$$

Combining (3.36), (3.35) and (3.43), we have, for $t \ge T$,

$$2a - bCt^{-\alpha_k} - bG_1\left(G_1^{-1}\Big(C(t-\kappa)^{-\alpha_k}\Big) + \frac{\ell(t)}{a-bC(t-\kappa)^{-\alpha_k}}\right) \le \ell'(t) \le 2a - bG_1\Big(\frac{\ell(t)}{a}\Big).$$

Consider the following ODE:

$$\overline{\ell}'(t) = 2a - bG_1\Big(\frac{\overline{\ell}(t)}{a}\Big), \quad \overline{\ell}(T) = \ell(T).$$

It is not hard to see that $\overline{\ell}(t)$ tends to ℓ_P exponentially. By comparison, for all $T \gg 1$,

$$\ell(t) - \ell_P \le \overline{\ell}(t) - \ell_P \le C_1 e^{-\rho t} \le \frac{C_2}{t^{\alpha_k}}, \quad t \ge T \tag{3.44}$$

for some positive constants ρ, C_1 and C_2. Hence, we obtain an upper estimate of $\ell(t) - \ell_P$.

We need to get a lower estimate of $\ell(t) - \ell_P$. The mean value theorem implies that

$$bG_1\left(z + \frac{\ell(t)}{a-bC(t-\kappa)^{-\alpha_k}}\right) = bG_1\left(z + \frac{\ell_P}{a-bC(t-\kappa)^{-\alpha_k}}\right) + \nu(t)(\ell(t) - \ell_P)$$

where

$$0 < \nu_1 := \left(g_1 - \frac{g_2}{g_3}\right)\frac{b}{a} \le \nu(t) \le \frac{2bg_1}{a}, \tag{3.45}$$

for $t \ge T$ because of (3.41) and (2.2). It follows that

$$\ell'(t) \ge Q(t) - \nu(t)(\ell(t) - \ell_P), \tag{3.46}$$

where

$$Q(t) := 2a - bCt^{-\alpha_k} - bG_1\left(G_1^{-1}\Big(C(t-\kappa)^{-\alpha_k}\Big) + \frac{\ell_P}{a-bC(t-\kappa)^{-\alpha_k}}\right).$$

By the l'Hospital rule, it is easy to obtain

$$\lim_{t\to\infty} \frac{Q(t)}{t^{-\alpha_k}} = -\delta$$

for some $\delta > 0$. Hence, there exists $K > 0$ such that $Q(t) \ge -Kt^{-\alpha_k}$ for all $t \ge T$. Together with (3.46), we have

$$(\ell(t) - \ell_P)' \ge -\frac{K}{t^{\alpha_k}} - \nu(t)(\ell(t) - \ell_P).$$

By multiplying both sides by $e^{\int_T^t \nu(s)ds}$ and integrating them over $[T,t]$, we obtain

$$\ell(t) - \ell_P \ge (\ell(T) - \ell_P)e^{-\int_T^t \nu(s)ds} - K\int_T^t \frac{1}{\tau^{\alpha_k}} e^{-\int_\tau^t \nu(s)ds} d\tau \tag{3.47}$$

for $t \geq T$ with sufficiently large T. Since $0 < \alpha_k < 1$ and

$$\begin{aligned}\int_T^t \frac{1}{\tau^{\alpha_k}} e^{-\int_\tau^t \nu(s)ds} d\tau &\leq \int_T^t \frac{1}{\tau^{\alpha_k}} e^{-\nu_1(t-\tau)} d\tau \\ &\leq \int_T^{(t+T)/2} \frac{1}{\tau^{\alpha_k}} e^{-\nu_1(t-T)/2} d\tau + \int_{(t+T)/2}^t \frac{2^{\alpha_k}}{(t+T)^{\alpha_k}} e^{-\nu_1(t-\tau)} d\tau \\ &\leq \frac{1}{(1-\alpha_k)t^{\alpha_k - 1}} e^{-\nu_1(t-T)/2} + \frac{2^{\alpha_k}}{\nu_1(t+T)^{\alpha_k}},\end{aligned}$$

where ν_1 is defined in (3.45), we obtain

$$\ell(t) - \ell_P \geq (\ell(T) - \ell_P)e^{-\nu_1(t-T)} - \frac{Kt^{1-\alpha_k}}{(1-\alpha_k)} e^{-\nu_1(t-T)/2} - \frac{2^{\alpha_k} K}{\nu_1(t+T)^{\alpha_k}} \geq -\frac{C_3}{t^{\alpha_k}}$$

with some positive constant C_3 when T is sufficiently large. Combining (3.47) and (3.44), we complete the proof. □

Lemma 3.15. *Assume that all hypothesis of Lemma 3.14 are satisfied and that $x'_{2k-2}(t) > 0$. Then*

$$\begin{aligned}&x'_{2k-1}(t) \to a \qquad \text{as } t \to \infty, \\ &x_{2k-1}(t) - x_{2k-2}(t) \geq \delta_1 \log t \qquad \text{as } t \to \infty, \\ &v(x_{2k-2}(t), t) = O(t^{-\widetilde{\alpha}_k}) \qquad \text{as } t \to \infty,\end{aligned}$$

where δ_1 is a positive constant and $0 < \widetilde{\alpha}_k < 1$.

Proof. Let $h(t) := x_{2k-1}(t) - x_{2k-2}(t)$. By simple calculations, we have

$$\begin{aligned}h'(t) &= x'_{2k-1}(t) - x'_{2k-2}(t) \\ &= bv(x_{2k-1}(t), t) - 2a + bv(x_{2k-2}(t), t).\end{aligned}$$

The term $v(x_{2k-1}(t), t)$ has been estimated in (3.43). To obtain an upper estimate of $v(x_{2k-2}(t), t)$ for large t, we divide our discussion into two steps.

Step 1: A lower estimate of h.

First, since $v_t = g(0, v) \geq -g_2 v/g_4$ for $x_{2k-2}(t) \leq x \leq x_{2k-1}(t)$ and $t \geq T^*$,

$$v(x_{2k-2}(t), t) \geq v\Big(x_{2k-2}(t), T_{2k-1}(x_{2k-2}(t))\Big) e^{-\frac{g_2}{g_4}\left(t - T_{2k-1}(x_{2k-2}(t))\right)} \tag{3.48}$$

for all large t, where $T_{2k-1}(z)$ is the arrival time of $x = x_{2k-1}(t)$ to z. As in the proof of (3.19), we see that

$$x'_{2k-1}(t) \to a \quad \text{as } t \to \infty. \tag{3.49}$$

By the mean value theorem, we see that

$$\frac{h(t)}{t - T_{2k-1}(x_{2k-2}(t))} = \frac{x_{2k-1}(t) - x_{2k-1}(T_{2k-1}(x_{2k-2}(t)))}{t - T_{2k-1}(x_{2k-2}(t))} \to a \quad \text{as } t \to \infty.$$

Thus there exists $\tau > T$ such that

$$a - \varepsilon < \frac{h(t)}{t - T_{2k-1}(x_{2k-2}(t))} < a + \varepsilon, \quad t \geq \tau. \tag{3.50}$$

By (3.48) and (3.50), we have

$$\begin{aligned}v(x_{2k-2}(t), t) &\geq v\Big(x_{2k-2}(t), T_{2k-1}(x_{2k-2}(t))\Big) e^{-\frac{g_2}{g_4}\left(\frac{h(t)}{a-\varepsilon}\right)} \\ &\geq C_1 e^{-\lambda h(t)},\end{aligned}$$

for some $C_1 > 0$ and $\lambda := g_2/[g_4(a-\varepsilon)]$. Together with (3.43) and the mean value theorem, we have

$$\begin{aligned} h'(t) &= bv(x_{2k-1}(t),t) - 2a + bv(x_{2k-2}(t),t) \\ &\geq bG_1\Big(\frac{\ell(t)}{a}\Big) - 2a + bC_1 e^{-\lambda h(t)} \\ &= bG_1\Big(\frac{\ell_P}{a}\Big) + \nu(t)(\ell(t)-\ell_P) - 2a + bC_1 e^{-\lambda h(t)} \\ &= \nu(t)(\ell(t)-\ell_P) + bC_1 e^{-\lambda h(t)} \end{aligned}$$

for some $\nu(t) > 0$ having a positive lower bound and $\ell(t) := x_{2k}(t) - x_{2k-1}(t)$. Then, by Lemma 3.14, we have

$$h'(t) \geq -\frac{\widehat{K}}{t^{\alpha_k}} + C_2 e^{-\lambda h(t)}, \quad t \geq \tau$$

for some positive constants $\widehat{K}$ and $C_2 := bC_1$.

Let $P(t) := e^{\lambda h(t)}$. Then we have

$$P'(t) \geq -\frac{\widehat{K}\lambda}{t^{\alpha_k}} P(t) + C_2\lambda, \quad t \geq \tau.$$

Since $0 < \alpha_k < 1$ and P is positive, for a sufficiently large constant τ, there are a constant $\delta \in (0, C_2/\widehat{K})$ such that

$$P(\tau) \geq \delta\tau^{\alpha_k}, \quad (C_2 - \widehat{K}\delta)\lambda > \alpha_k\delta\tau^{\alpha_k - 1}.$$

Then

$$(P(t) - \delta t^{\alpha_k})' \geq -\frac{\widehat{K}\lambda}{t^{\alpha_k}}(P - \delta t^{\alpha_k}) + C_2\lambda - \widehat{K}\lambda\delta - \alpha_k\delta t^{\alpha_k-1} \geq -\frac{\widehat{K}\lambda}{t^{\alpha_k}}(P - \delta t^{\alpha_k})$$

for any $t \geq \tau$. Thus $P(t) \geq \delta t^{\alpha_k}$ for all $t \geq \tau$. Thus, there is a positive constant C_3 such that

$$h(t) \geq \delta_1 \log t - C_3 \tag{3.51}$$

for all large t, where

$$\delta_1 := \frac{\alpha_k}{\lambda} = \frac{\alpha_k g_4(a-\varepsilon)}{g_2}.$$

Thus **Step 1** is completed.

Step 2: An upper estimate of v.

By (3.43), we have

$$0 \leq v(x,t) \leq v(x_{2k-1}(t),t) \leq M^* \qquad \text{for } x_{2k-2}(t) \leq x \leq x_{2k-1}(t),\ t \geq T^*$$

with a sufficiently large constant T^* where $M^* := \max\{4b/a, M\}$. Since $v_t = g(0,v)$ for $x_{2k-2}(t) \leq x \leq x_{2k-1}(t)$ and $t \geq T^*$,

$$0 \leq v(x_{2k-2}(t),t) \leq v\Big(x_{2k-2}(t), T_{2k-1}(x_{2k-2}(t))\Big) e^{-\frac{g_2}{g_3M^*+g_4}\Big(t - T_{2k-1}(x_{2k-2}(t))\Big)}$$

for all large t. Then there exists a positive constant C_4 such that

$$0 \leq v(x_{2k-2}(t),t) \leq C_4 e^{-\mu(t - T_{2k-1}(x_{2k-2}(t)))}$$

for all large t, where $\mu := g_2/(g_3M^* + g_4)$. By (3.50),

$$0 \leq v(x_{2k-2}(t),t) \leq C_4 e^{-\hat{\mu}h(t)}$$

for some $\hat{\mu} := \mu/(a+\varepsilon) > 0$ and for all large t. Together with (3.51), we get

$$v(x_{2k-2}(t), t) = O(t^{-\widetilde{\alpha}_k}) \qquad \text{as } t \to \infty, \tag{3.52}$$

where

$$\widetilde{\alpha}_k := \hat{\mu}\delta_1 = \frac{\alpha_k g_4(a-\varepsilon)}{(g_3 M^* + g_4)(a+\varepsilon)} \in (0,1).$$

Then **Step 2** is completed.

By combining (3.49), (3.51) and (3.52), the proof of Lemma 3.15 is completed. □

By the similar argument as in Lemma 3.14 and Lemma 3.15, we have the following two lemmas.

Lemma 3.16. *Assume the same hypothesis of Proposition 3.13. If there exist constants $C > 0$ and $\alpha_k > 0$ such that*

$$0 \le v(x_{2k}(t), t) \le Ct^{-\alpha_k}, \quad t \ge T^*$$

for some $k \in \{-1, -2, \cdots, -m_-\}$, then

$$x_{2k+1}(t) - x_{2k}(t) = \ell_P + O(t^{-\alpha_k}) \text{ as } t \to \infty.$$

Lemma 3.17. *Assume that all hypothesis of Lemma 3.16 are satisfied and that $x'_{2k+2}(t) < 0$. Then*

$$\begin{aligned} &x'_{2k+1}(t) \to -a \qquad \text{as } t \to \infty, \\ &x_{2k+2}(t) - x_{2k+1}(t) \ge \delta_k \log t \qquad \text{as } t \to \infty, \\ &v(x_{2k+2}(t), t) = O(t^{-\alpha_k}) \qquad \text{as } t \to \infty, \end{aligned}$$

where δ_k and α_k are positive constants.

By Lemmas 3.1-3.2 and Lemmas 3.14-3.17, we prove Proposition 3.13 as follows:

Proof of Proposition 3.13. We start with $j = m_+$. Since $x = x_{2m_+}(t)$ is the right-most interface with $x'_{2m_+}(t) > 0$, by Lemma 3.1(iv), $x'_{2m_+}(t) \to a$ as $t \to \infty$. Also, by Lemma 3.1(i), there exist constants $C_{m_+} > 0$ and $\alpha_{m_+} \in (0,1)$ such that (3.35) holds replacing k, C and α_k by m_+, C_{m_+} and α_{m_+}, respectively. Hence, Lemma 3.14 can be applied to obtain (3.29) with $j = m_+$. Next, by Lemma 3.15, we obtain (3.30) with $j = m_+$ and (3.34) with $k = 2m_+ - 1$. In addition, Lemma 3.15 also gives $v(x_{2m_+-2}(t), t) = O(t^{-\alpha_{m_+}})$ for some $\alpha_{m_+} \in (0,1)$, which allows us to apply Lemma 3.14 with $k = m_+ - 1$. By repeating this process for $j = m_+ - 1, \cdots, 1$ sequentially, we can derive (3.29), (3.30), (3.33) with positive index and (3.34) with positive k.

Similarly, we can show (3.31), (3.32) and (3.33) with negative index and (3.34) with negative k using Lemma 3.2, Lemma 3.16 and Lemma 3.17. This completes the proof. □

We are ready to show Theorem 2.6.

Proof of Theorem 2.6. From Proposition 3.11 and Remark 3.12, we may assume that (Ω, v) is a classical solution for $t \ge T$ (for some large T) with the word $\mathbf{w} := \mathrm{SGN}[(\Omega(T), v(\cdot, T))]$ is of type (X) for X=I, II, III. As mentioned in Remark 3.12, if $\mathbf{w}$ is of type (I), clearly, the solution is of type (I). Namely, the conclusion (I) of Theorem 2.6 holds.

Next, we consider the case where $\mathbf{w}$ is of type (II), i.e., (3.27) holds. By Proposition 3.13, we know that (2.11), (2.12), (2.14), (2.15) and (2.13) hold. So we only need to show (2.16). Recall that $T_i(y)$ is the arrival time of the interface $x = x_i(t)$ ($i = -2m_-, ..., -1, 1, .., 2m_+$) to y. Note that the numbering of interfaces is different from problem (2.3).

Now, by using G_0 and G_1, we can compute the function $v(x,t)$ as follows:

(i) For $x \geq x_{2m_+}(t)$ and $t \geq T$,

$$v(x,t) = G_0\left(G_0^{-1}(v(x,T)) + t - T\right).$$

(ii) For $x_{2k-1}(t) \leq x < x_{2k}(t)$ and $t \geq T_{2k-1}(x_{2k}(T))$ with $k = 1, ..., m_+$,

$$v(x,t) = G_1\left(G_1^{-1}(v(x, T_{2k}(x))) + t - T_{2k}(x)\right).$$

(iii) For $x_{2k-2}(t) \leq x < x_{2k-1}(t)$ and $t \geq T_{2k-2}(x_{2k-1}(T))$ with $k = 2, ..., m_+$,

$$v(x,t) = G_0\left(G_0^{-1}(v(x, T_{2k-1}(x))) + t - T_{2k-1}(x)\right).$$

(iv) For $x_{-1}(t) < x < x_1(t)$,

$$v(x,t) = \begin{cases} G_0\left(G_0^{-1}(v(x,T_1(x))) + t - T_1(x)\right) & \text{for } x_1(T) \leq x < x_1(t),\ t \geq T_1(x), \\ G_0\left(G_0^{-1}(v(x,T)) + t - T\right) & \text{for } x_{-1}(T) < x < x_1(T),\ t \geq T, \\ G_0\left(G_0^{-1}(v(x,T_{-1}(x))) + t - T_{-1}(x)\right) & \text{for } x_{-1}(t) < x \leq x_{-1}(T),\ t \geq T_{-1}(x). \end{cases}$$

(v) For $x \leq x_{-2m_-}(t)$ and $t \geq T$,

$$v(x,t) = G_0\left(G_0^{-1}(v(x,T)) + t - T\right).$$

(vi) For $x_{2k}(t) < x \leq x_{2k+1}(t)$ and $t \geq T_{2k+1}(x_{2k}(T))$ with $k = -m_-, ..., -1$,

$$v(x,t) = G_1\left(G_1^{-1}(v(x, T_{2k}(x))) + t - T_{2k}(x)\right).$$

(vii) For $x_{2k+1}(t) < x \leq x_{2k+2}(t)$ and $t \geq T_{2k+2}(x_{2k+1}(T))$ with $k = -m_-, ..., -2$,

$$v(x,t) = G_0\left(G_0^{-1}(v(x, T_{2k+1}(x))) + t - T_{2k+1}(x)\right).$$

By (2.17) and the form of φ_P given in Theorem 2.1, $v_\infty(x,t)$ can be represented by G_0 and G_1 as follows:

(1) For $x \geq x_{2m_+}(t)$ and $t \geq T$,

$$v_\infty(x,t) = \sum_{j=-m_-}^{-1} G_0\left(-\frac{x_{2j}(t) - x + \ell_P}{a} + G_0^{-1}\left(\frac{2a}{b}\right)\right) := \mathcal{G}^-(x,t).$$

(2) For $x_{2m_+}(t) - \ell_P \leq x < x_{2m_+}(t)$,

$$v_\infty(x,t) = G_1\left(\frac{x_{2m_+}(t) - x}{a}\right) + \mathcal{G}^-(x,t).$$

(3) For $x_{2k-2}(t) \leq x < x_{2k}(t) - \ell_P$, $k = 2, ..., m_+$,

$$v_\infty(x,t) \quad = \quad \mathcal{G}^-(x,t) + \sum_{j=k}^{m_+} G_0\left(-\frac{x - x_{2j}(t) + \ell_P}{a} + G_0^{-1}\left(\frac{2a}{b}\right)\right).$$

(4) For $x_{2k-2}(t) - \ell_P \leq x < x_{2k-2}(t)$, $k = 2, ..., m_+$,

$$v_\infty(x,t) \quad = \quad G_1\left(\frac{x_{2k-2}(t) - x}{a}\right) + \mathcal{G}^-(x,t) + \sum_{j=k}^{m_+} G_0\left(-\frac{x - x_{2j}(t) + \ell_P}{a} + G_0^{-1}\left(\frac{2a}{b}\right)\right).$$

(5) For $x_{-2}(t) + \ell_P < x < x_2(t) - \ell_P$,

$$v_\infty(x,t) \quad = \quad \mathcal{G}^-(x,t) + \sum_{j=1}^{m_+} G_0\left(-\frac{x - x_{2j}(t) + \ell_P}{a} + G_0^{-1}\left(\frac{2a}{b}\right)\right).$$

(6) For $x \le x_{-2m_-}(t)$,

$$v_\infty(x,t) = \sum_{j=1}^{m_+} G_0\left(-\frac{x - x_{2j}(t) + \ell_P}{a} + G_0^{-1}\left(\frac{2a}{b}\right)\right) := \mathcal{G}^+(x,t).$$

(7) For $x_{-2m_-}(t) < x \le x_{-2m_-}(t) + \ell_P$,

$$v_\infty(x,t) = G_1\left(\frac{x - x_{-2m_-}(t)}{a}\right) + \mathcal{G}^+(x,t).$$

(8) For $x_{2k}(t) - \ell_P < x \le x_{2k+2}(t)$, $k = -m_-, .., -2$,

$$v_\infty(x,t) \;=\; \sum_{j=-m_-}^{k} G_0\left(-\frac{x_{2j}(t) - x + \ell_P}{a} + G_0^{-1}\left(\frac{2a}{b}\right)\right) + \mathcal{G}^+(x,t).$$

(9) For $x_{2k+2}(t) < x \le x_{2k+2}(t) + \ell_P$, $k = -m_-, .., -2$,

$$v_\infty(x,t) \;=\; G_1\left(\frac{x - x_{2k+2}(t)}{a}\right) + \mathcal{G}^+(x,t) + \sum_{j=-m_-}^{k} G_0\left(-\frac{x_{2j}(t) - x + \ell_P}{a} + G_0^{-1}\left(\frac{2a}{b}\right)\right).$$

Later we will use the following fact several times:

$$\lim_{t\to+\infty} \sup_{x \ge x_1(t)} \left|\mathcal{G}^-(x,t)\right| = 0. \tag{3.53}$$

Indeed, in view of (2.11), we have

$$x_{2j}(t) - x \le x_{2j}(t) - x_1(t) \to -\infty \quad \text{as } t \to +\infty$$

for $x \ge x_1(t)$ and $j = -m_-, ..., -1$. Together with the fact $G_0(+\infty) = 0$, we see that (3.53) is true. Similarly, we can obtain

$$\lim_{t\to+\infty} \sup_{x \le -x_1(t)} \left|\mathcal{G}^+(x,t)\right| = 0. \tag{3.54}$$

From (i) and (1) we see that

$$\begin{aligned}
&\lim_{t\to+\infty} \sup_{x \ge x_{2m_+}(t)} |v(x,t) - v_\infty(x,t)| \\
&\le \lim_{t\to+\infty} \sup_{x \ge x_{2m_+}(t)} \left|G_0\left(G_0^{-1}(v(x,T)) + t - T\right)\right| + \lim_{t\to+\infty} \sup_{x \ge x_{2m_+}(t)} \left|\mathcal{G}^-(x,t)\right|.
\end{aligned}$$

By $G_0(+\infty) = 0$,

$$\lim_{t\to+\infty} \sup_{x \ge x_{2m_+}(t)} \left|G_0\left(G_0^{-1}(v(x,T)) + t - T\right)\right| = 0.$$

Together with (3.53) and $\{x \ge x_{2m_+}(t)\} \subset \{x \ge x_1(t)\}$, we then obtain

$$\lim_{t\to+\infty} \sup_{x \ge x_{2m_+}(t)} |v(x,t) - v_\infty(x,t)| = 0. \tag{3.55}$$

From (ii), (2) and (2.15), we see that

$$\begin{aligned}
&\lim_{t\to+\infty} \sup_{x_{2m_+-1}(t) \le x \le x_{2m_+}(t)} |v(x,t) - v_\infty(x,t)| \\
&\le \lim_{t\to+\infty} \sup_{x_{2m_+-1}(t) \le x \le x_{2m_+}(t)} \Bigg|G_1\left(G_1^{-1}(v(x,T_{2m_+}(x))) + t - T_{2m_+}(x)\right) \\
&\qquad - G_1\Big(\frac{x_{2m_+}(t) - x}{a}\Big)\Bigg| + \lim_{t\to+\infty} \sup_{x_{2m_+-1}(t) \le x \le x_{2m_+}(t)} \left|\mathcal{G}^-(x,t)\right|.
\end{aligned}$$

By the similar argument used in Lemma 3.4, we have

$$\lim_{t\to+\infty}\sup_{x_{2m_+-1}(t)\le x\le x_{2m_+}(t)}\left|G_1\left(G_1^{-1}(v(x,T_{2m_+}(x)))+t-T_{2m_+}(x)\right)-G_1\Big(\frac{x_{2m_+(t)}-x}{a}\Big)\right|=0.$$

Together with (2.12), (3.53) and $\{x\ge x_{2m_+-1}(t)\}\subset\{x\ge x_1(t)\}$, we then obtain

$$\lim_{t\to+\infty}\sup_{x_{2m_+-1}(t)\le x\le x_{2m_+}(t)}|v(x,t)-v_\infty(x,t)|=0. \tag{3.56}$$

Given $k\in\{2,...,m_+\}$, from (iii), (3) and (2.15), we see that

$$\begin{aligned}
&\lim_{t\to+\infty}\sup_{x_{2k-2}(t)\le x<x_{2k-1}(t)}|v(x,t)-v_\infty(x,t)|\\
\le\;&\lim_{t\to+\infty}\sup_{x_{2k-2}(t)\le x<x_{2k-1}(t)}\Big|G_0\left(G_0^{-1}(v(x,T_{2k-1}(x)))+t-T_{2k-1}(x)\right)\\
&\qquad -G_0\left(-\frac{x-x_{2k}(t)+\ell_P}{a}+G_0^{-1}\left(\frac{2a}{b}\right)\right)\Big|\\
&+\lim_{t\to+\infty}\sup_{x_{2k-2}(t)\le x<x_{2k-1}(t)}\left|\mathcal{G}^-(x,t)\right|\\
&+\lim_{t\to+\infty}\sup_{x_{2k-2}(t)\le x<x_{2k-1}(t)}\left|\sum_{j=k}^{m_+}G_0\left(-\frac{x-x_{2j}(t)+\ell_P}{a}+G_0^{-1}\left(\frac{2a}{b}\right)\right)\right|\\
=:\;&J_1+J_2+J_3.
\end{aligned}$$

By the similar argument used in Lemma 3.5, we have $J_1=0$. By (3.53), $J_2=0$. For J_3, using (2.14), we have $x-x_{2j}(t)\to-\infty$ for $j=k+1,\cdots m_+$. Then, using $G_0(+\infty)=0$, it follows that $J_3=0$. Hence, we have

$$\lim_{t\to+\infty}\sup_{x_{2k-2}(t)\le x<x_{2k-1}(t)}|v(x,t)-v_\infty(x,t)|=0. \tag{3.57}$$

Given $k\in\{1,...,m_+-1\}$, by (ii) and (4) replaced k by $k+1$, we obtain

$$\begin{aligned}
&\lim_{t\to+\infty}\sup_{x_{2k-1}(t)\le x<x_{2k}(t)}|v(x,t)-v_\infty(x,t)|\\
\le\;&\lim_{t\to+\infty}\sup_{x_{2k-1}(t)\le x<x_{2k}(t)}\left|G_1\left(G_1^{-1}(v(x,T_{2k}(x)))+t-T_{2k}(x)\right)-G_1\left(\frac{x_{2k}(t)-x}{a}\right)\right|\\
&+\lim_{t\to+\infty}\sup_{x_{2k-1}(t)\le x<x_{2k}(t)}\left|\mathcal{G}^-(x,t)\right|\\
&+\lim_{t\to+\infty}\sup_{x_{2k-1}(t)\le x<x_{2k}(t)}\left|\sum_{j=k+1}^{m_+}G_0\left(-\frac{x-x_{2j}(t)+\ell_P}{a}+G_0^{-1}\left(\frac{2a}{b}\right)\right)\right|.
\end{aligned}$$

Then, using the process used in deriving (3.56) and $J_3=0$, we can have

$$\lim_{t\to+\infty}\sup_{x_{2k-1}(t)\le x<x_{2k}(t)}|v(x,t)-v_\infty(x,t)|=0. \tag{3.58}$$

Also, by (iv) and (5) we can prove

$$\lim_{t\to+\infty}\sup_{x_{-1}(t)<x<x_1(t)}|v(x,t)-v_\infty(x,t)|=0. \tag{3.59}$$

since $\lim_{t\to+\infty}\sup_{x_{-1}(t)<x<x_1(t)}|v(x,t)|=0$ and $\lim_{t\to+\infty}\sup_{x_{-1}(t)<x<x_1(t)}|v_\infty(x,t)|=0$.

By using the same argument as above, we can show

$$\lim_{t\to+\infty}\sup_{x\le x_{-1}(t)}|v(x,t)-v_\infty(x,t)|=0. \tag{3.60}$$

Combining (3.55)-(3.60), we have proved (2.16) when $\mathbf{w}$ is of type (II). Therefore, the solution is of type (II) when the word $\mathbf{w}$ is of type (II).

Finally, we claim the classical solution (Ω, v) of (2.3) for $t\ge T$ is of type (III) when $\mathbf{w}$ is of type (III), i.e., (3.28) holds. By Proposition 3.13, we know that (2.11)-(2.13) and (2.18) hold. So we only need to show (2.19). Similarly, we define $T_0^+(y)$ and $T_0^-(y)$ are the arrival time of the interface $x=y_1(t)$ to y and $x=y_{-1}(t)$ to y, respectively. Now we compute the function $v(x,t)$ for $t\ge T$ as in the previous case. In fact, (i), (ii), (iii), (v), (vi) and (vii) given in the above still hold in this case. For the remaining region,

(iv-1) For $y_1(t)\le x<x_1(t)$ and $t\ge T_0^+(x_1(T))$, simple calculations give

$$v(x,t)=G_0\left(G_0^{-1}(v(x,T_1(x)))+t-T_1(x)\right).$$

(iv-2) for $y_{-1}(t)<x<y_1(t)$,

$$v(x,t)=\begin{cases} G_1\left(G_1^{-1}(v(x,T_0^+(x)))+t-T_0^+(x)\right), & \text{for } y_1(T)\le x<y_1(t),\ t\ge T_0^+(x),\\ G_1\left(G_1^{-1}(v(x,T))+t-T\right), & \text{for } y_{-1}(T)<x<y_1(T),\ t\ge T,\\ G_1\left(G_1^{-1}(v(x,T_0^-(x)))+t-T_0^-(x)\right), & \text{for } y_{-1}(t)<x\le y_{-1}(T),\ t\ge T_0^-(x).\end{cases}$$

(iv-3) For $x_{-1}(t)<x\le y_{-1}(t)$ and $t\ge T_0^-(x_{-1}(T))$,

$$v(x,t)=G_0\left(G_0^{-1}(v(x,T_{-1}(x)))+t-T_{-1}(x)\right).$$

For $x\ge x_2(t)-\ell_P$, it is easy to check $v_\infty(x,t)$ is represented in (1)-(4); while $v_\infty(x,t)$ is represented in (5) if $y_1(t)\le x<x_2(t)-\ell_P$ and $x_{-2}(t)+\ell_P<x\le y_{-1}(t)$. When $x\le x_{-2}(t)+\ell_P$, $v_\infty(x,t)$ is represented in (6)-(9). By using the argument in deriving (2.16), we can obtain

$$\lim_{t\to+\infty}\sup_{x\ge y_1(t)}|v(x,t)-v_\infty(x,t)|=\lim_{t\to+\infty}\sup_{x\le y_{-1}(t)}|v(x,t)-v_\infty(x,t)|=0. \tag{3.61}$$

Finally, we need to show that

$$\lim_{t\to+\infty}\sup_{x\in(y_{-1}(t),y_{-1}(t)+K]}|v(x,t)-v_\infty(x,t)|=\lim_{t\to+\infty}\sup_{x\in(y_1(t)-K,y_1(t)]}|v(x,t)-v_\infty(x,t)|=0. \tag{3.62}$$

Notice that

$$\begin{aligned} v_\infty(x,t) &= G_1\left(\frac{y_1(t)-x}{a}\right)+\mathcal{G}^-(x,t)+\mathcal{G}^+(x,t), \quad y_1(t)-K\le x<y_1(t),\\ v_\infty(x,t) &= G_1\left(\frac{x-y_{-1}(t)}{a}\right)+\mathcal{G}^-(x,t)+\mathcal{G}^+(x,t), \quad y_{-1}(t)<x\le y_{-1}(t)+K.\end{aligned}$$

Then, by (iv-3), (3.53), (3.54) and using the similar argument as in the proof of Lemma 3.4, we can derive (3.62). In view of (3.61) and (3.62), we see that (2.19) holds. Therefore, the proof of Theorem 2.6 is completed. □

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References

[1] S. Alonso, F. Sagues and A. S. Mikhailov, *Taming Winfree turbulence of scroll waves in excitable media*, Science **299** (2003) 1722–1725,

[2] E. Ben-Jacob, P. Garik, *The formation of patterns in non-equilibrium growth*, Nature 343.6258 (1990) 523–530

[3] E. Ben-Jacob, O. Schochet, A. Tenenbaum, I. Cohen, A. Czirok and T. Vicsek *Generic modelling of cooperative growth patterns in bacterial colonies*, Nature 368.6466 (1994) 46–49

[4] X.Y. Chen, *Dynamics of interfaces in reaction diffusion systems*. Hiroshima Mathematical Journal **21** (1991) 47–83.

[5] X.F. Chen, *Generation and propagation of interfaces for reaction-diffusion systems*, Trans. Amer. Math. Soc. **334** (1992) 877–913.

[6] X.F. Chen, C. Gao, *Well-posedness of a free boundary problem in the limit of slow-diffusion fast-reaction systems*. J. Partial Differ. Equ 19 (2006) 48–79.

[7] Y.-Y. Chen, Y. Kohsaka and H. Ninomiya, *Traveling spots and traveling fingers in singular limit problems of reaction-diffusion systems*, Discrete Cont. Dyn. Syst. (Ser. B) **19** (2014), 697–714.

[8] Y.-Y. Chen, H. Ninomiya and R. Taguchi, *Traveling spots on multi-dimensional excitable media*, Journal of Elliptic and Parabolic Equations **1** (2015), 281–305.

[9] Y.-Y. Chen, H. Ninomiya and C.-H. Wu, *Global existence and uniqueness of solutions for one-dimensional reaction-interface systems*, submitted. See also at https://arxiv.org/abs/2104.04971.

[10] A. Ducrot, T. Giletti and H. Matano, *Existence and convergence to a propagating terrace in one-dimensional reaction-diffusion equations*, Transactions of the American Mathematical Society **366** (2014) 5541–5566.

[11] P.C. Fife, *Propagator-controller systems and chemical patterns*. in *"Non-equilibrium dynamics in chemical systems"* edts. by C. Vidal and A. Pacault (1984), 76-88.

[12] Y. Giga, S. Goto and H. Ishii,*Global existence of weak solutions for interface equations coupled with diffusion equations*, SIAM Journal on Mathematical Analysis 23.4 (1992) 821–835.

[13] A. Hagberg, E. Meron, *Pattern formation in non-gradient reaction-diffusion systems: the effects of front bifurcations*, Nonlinearity, **7** (1994) 805.

[14] D. Hilhorst, Y. Nishiura, and M. Mimura, *A free boundary problem arising in some reacting-diffusing system*. Proceedings of the Royal Society of Edinburgh: Section A Mathematics 118.3-4 (1991): 355–378.

[15] R. Kapral,K. Showalter (eds.), *Chemical waves and patterns*, Vol. 10. (2012) Springer Science & Business Media.

[16] K. Krischer and A. Mikhailov, *Bifurcation to traveling spots in reaction-diffusion systems*, Physical Review Letters **73** (1994), 3165–3168.

[17] E. Meron, *Pattern formation in excitable media*, Phys. Rep. **218** (1992) 1–66.

[18] H. Ninomiya, C.-H. Wu, *Traveling curved waves in two dimensional excitable media*, SIAM Journal on Mathematical Analysis **49** (2017), 777–817.

[19] L. M. Pismen, *Nonlocal boundary dynamics of traveling spots in a reaction-diffusion system*, Physical Review Letters **86** (2001), 548–551.

[20] J. Rauch, J. Smoller, *Qualitative theory of the FitzHugh-Nagumo equations*, Advances in Math. **27** (1978), 12–44.

[21] J. Rinzel, J. B. Keller, *Traveling wave solutions of a nerve conduction equation*, Biophysical Journal **13** (1973), 1313–1337.

[22] J.J. Tyson, J.P. Keener, *Singular perturbation theory of traveling waves in excitable media (a review)*, Physica D **32** (1988), 327–361.

[23] A.T. Winfree, *Spiral waves of chemical activity*, Science. **175** (1972), 634–636.

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